2018 MATH1010
Lecture 5: Trigonometric function
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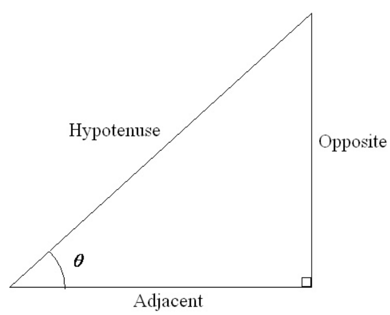
The lecture note was used during 2016-17 Term 1. It is for reference only. It may contain typos. Read at your own risk.

1 Basic properties

1. **Definition**

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}},$$
$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}},$$
$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}.$$



2. **radian**

   - In secondary school, when we use the trigonometric functions, we use **degree** in the parameter. So we write $\cos 18°$, $\tan 77°$.
   - However, in higher mathematics, we use **radians** instead of degree. The conversion is given by

   $$\pi = 180°$$

   or

   $$1° = \frac{\pi}{180}.$$

- So instead of writing $\cos 18°$, we write $\cos \frac{\pi}{10}$.
- The radian is so chosen such that the arc length of a sector (see picture below) is $L = r\theta$. Also, the area is $A = \frac{r^2\theta}{2}$

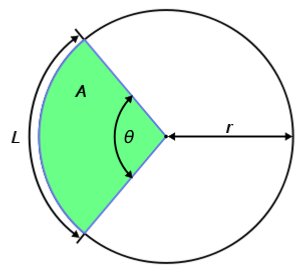


Figure 1: $L = r\theta$, $A = \frac{r^2\theta}{2}$

- Some sample values.

Angles in degree	0°	30°	45°	60°	90°	180°	360°
Angles in radian	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	2π

- There is no unit for radians. In the trigonometric functions, when we see no unit is the given parameter, it is understood that it means radians.
  No degree = radian

3. The graph of sin, cos, tan.

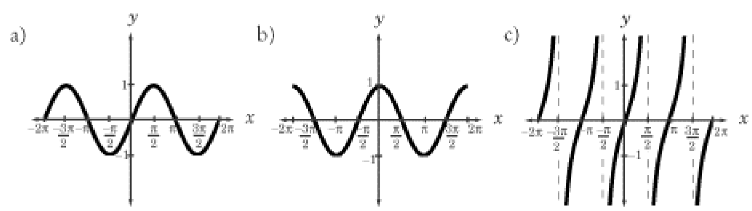


Figure 2: a) $y = \sin(x)$ b) $y = \cos(x)$ c) $y = \tan(x)$

4. Some simple identities.

$$\tan(\theta + \pi) = \tan\theta,$$

$$\sin(\theta + 2\pi) = \sin\theta,$$

$$\cos(\theta + 2\pi) = \cos\theta,$$

$$\sin(\frac{\pi}{2} - \theta) = \cos\theta,$$
$$\cos(\frac{\pi}{2} - \theta) = \sin\theta,$$
$$\cos(-\theta) = \cos\theta,$$
$$\sin(-\theta) = -\sin\theta,$$
$$\cos^2\theta + \sin^2\theta = 1,$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

5. **Addition formulas**

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi, \tag{1}$$
$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi. \tag{2}$$

Proof skipped.

How to memorize the formula (can be skipped, but helpful if you understand it).
Formally, we can write (this is called the Euler's formula, but let's take it formally.

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Here $e = 2.7182818...$ is a constant we will define it later.

$$e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi}.$$

So

$$\cos(\theta + \phi) + i\sin(\theta + \phi) = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$$
$$= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi).$$

Then compare the real part and the imaginary part.

6. **Double angle formulas**

$$\cos 2\theta = \cos^2\theta - \sin^2\theta,$$
$$\sin 2\theta = 2\sin\theta\cos\theta.$$

Proof. Set $\phi = \theta$ in (1) and (2) □

7. **Half-angle formulas**

$$\cos^2\theta = \frac{1+\cos 2\theta}{2},$$
$$\sin^2\theta = \frac{1-\cos 2\theta}{2}.$$

Proof. By the double angle formula

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin\theta.$$

The result then follows easily. □

8. **Product-to-sum formula**

$$\cos\theta\cos\phi = \frac{\cos(\theta-\phi)+\cos(\theta+\phi)}{2},$$
$$\sin\theta\sin\phi = \frac{\cos(\theta-\phi)-\cos(\theta+\phi)}{2},$$
$$\cos\theta\sin\phi = \frac{\sin(\theta+\phi)-\sin(\theta-\phi)}{2}.$$

Proof. We will prove the first formula, the other formulas can be proved similarly. By (1)

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi.$$

Replace ϕ by $-\phi$ we obtain

$$\cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi.$$

Summing up both formulas, we have

$$\cos(\theta+\phi) + \cos(\theta-\phi) = 2\cos\theta\cos\phi.$$

□

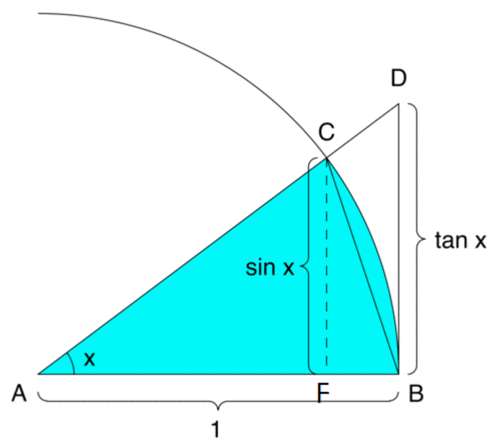
2 $\frac{\sin x}{x}$

Theorem 1 *We have*

1. $-|x| \le \sin x \le |x|$

2. $-|x| \leq 1 - \cos x \leq |x|$

Proof.



The arc length BC $\geq$ length of of BC $=$ the square root of FC^2+FB^2. Taking square on both sides.

$$x^2 \geq \sin^2 x + (1 - \cos x)^2.$$

So

$$x^2 \geq \sin^2 x$$

and

$$x^2 \geq (1 - \cos x)^2.$$

Taking square roots on both sides, we have

$$|x| \geq |\sin x|$$

and

$$|x| \geq |1 - \cos x|.$$

The result follows easily. ☐

Corollary 2

$$\lim_{x \to 0} \sin x = 0,$$

$$\lim_{x \to 0} \cos x = 1.$$

Proof. By the previous proposition

$$-|x| \le \sin x \le |x|$$

Because

$$\lim_{x\to 0} |x| = \lim_{x\to 0}(-|x|) = 0.$$

By the squeeze theorem,

$$\lim_{x\to 0} \sin x = 0.$$

Similarly

$$-|x| \le 1 - \cos x \le |x|.$$

Again, by the squeeze theorem

$$0 = \lim_{x\to 0}(1 - \cos x) = 1 - \lim_{x\to 0} \cos x.$$

The result follows. □

Theorem 3

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Proof. We assume $x > 0$. Refer to the picture in the proof of Theorem 1.

Area of sector ABC $\le$ Area of △ ABD.

So

$$\frac{x}{2} \le \frac{\tan x}{2} \tag{3}$$

Combining with the inequality $\sin x \le x$.

$$x \cos x \le \sin x \le x,$$

i.e.,

$$\cos x \le \frac{\sin x}{x} \le 1.$$

Because

$$\lim_{x\to 0^+} \cos x = 1$$

and

$$\lim_{x\to 0^+} 1 = 1.$$

By the squeeze theorem

$$\lim_{x\to 0^+} \frac{\sin x}{x} = 1.$$

Because $\frac{\sin x}{x}$ is an even function, i.e.

$$\frac{\sin x}{x} = \frac{\sin(-x)}{-x}.$$

So the left hand limit as x tends to 0 is the same as the right hand limit as x tends to 0. So

$$\lim_{x\to 0^+} \frac{\sin x}{x} = \lim_{x\to 0^-} \frac{\sin x}{x} = 1.$$

The result follows easily.

$\square$

Example 1 Compute $\lim\limits_{x\to 0} \dfrac{\sin 3x}{x}$.
Answer:

$$\begin{aligned}\lim_{x\to 0} \frac{\sin 3x}{x} &= \lim_{x\to 0} 3 \times \frac{\sin 3x}{3x} \\ &= 3 \lim_{y\to 0} \frac{\sin y}{y} \\ &= 3 \times 1 = 3.\end{aligned}$$

Example 2 Compute $\lim\limits_{x\to 0} \dfrac{\sin 2x}{\sin 3x}$
Answer:

$$\begin{aligned}\lim_{x\to 0} \frac{\sin 2x}{\sin 3x} &= \lim_{x\to 0} \frac{2}{3} \frac{\frac{\sin 2x}{2x}}{\frac{\sin 3x}{3x}} \\ &= \frac{2}{3} \frac{\lim_{x\to 0} \frac{\sin 2x}{2x}}{\lim_{x\to 0} \frac{\sin 3x}{3x}} \\ &= \frac{2}{3} \frac{\lim_{y\to 0} \frac{\sin y}{y}}{\lim_{z\to 0} \frac{\sin z}{z}} \\ &= \frac{2}{3}\end{aligned}$$

Example 3 Compute $\lim_{x\to 0}\dfrac{\tan x}{x}$
Answer:

$$\begin{aligned}\lim_{x\to 0}\frac{\tan x}{x} &= \lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{1}{\cos x}\\ &= 1\times 1 = 1.\end{aligned}$$

Example 4 Compute $\lim_{x\to 0}\dfrac{1-\cos x}{x^2}$
Answer: By the half angle formula

$$1-\cos x = 2\sin^2\frac{x}{2}.$$

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos x}{x^2} &= 2\left(\lim_{x\to 0}\frac{\sin x/2}{x}\right)^2\\ &= 2\times\left(\frac{1}{2}\right)^2 = \frac{1}{2}.\end{aligned}$$

Remark: Later in this course, you will learn a more powerful method called **L'Hopital's rule** to deal with limit in the form of

$$\lim_{x\to c}\frac{f(x)}{g(x)}$$

with $f(c) = g(c) = 0$. The method you learn here will become useless.