2018 MATH1010
Lecture 4: Limit II
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The lecture note was used during 2016-17 Term 1. It is for reference only. It may contain typos. Read at your own risk.

1 Relation between Limits of sequence and functions

Theorem 1 $\lim_{x\to c} f(x) = L$ *if and only if for every sequence* $\{a_n\}$ *such that* $a_n \neq c$ *for every* n *and*

$$\lim_{n\to\infty} a_n = c,$$

then

$$\lim_{n\to\infty} f(a_n) = L.$$

We can use the theorem to show that $\lim_{x\to c} f(x)$ doesn't exist:

1. Method 1: find a sequence $\{a_n\}$, $a_n \neq c$ for every n and $\lim_{n\to\infty} a_n = c$ but $\lim_{n\to\infty} f(a_n)$ does not exist.
2. Method 2: find two sequences $\{a_n\}$, $\{b_n\}$, $a_n \neq c$, $b_n \neq c$ for every n and $\lim_{n\to\infty} a_n = c$, $\lim_{n\to\infty} b_n = c$ but
$$\lim_{n\to\infty} f(a_n) \neq \lim_{n\to\infty} f(b_n).$$

Example 2 *Let* $f : \mathbf{R} \to \mathbf{R}$ *defined by*

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational, i.e., } x = \frac{p}{q} \text{ for some integers, } p, q \\ 0 & \text{if } x \text{ is not rational} \end{cases}$$

We want to show that $\lim_{n\to\infty} f(x)$ *does not exists.*
Let $a_n = \frac{1}{n}$, $b_n = \frac{\sqrt{2}}{n}$. *Then* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$. *Because* a_n *is rational, so* $f(a_n) = 1$. *Since* b_n *is not rational, so* $f(b_n) = 0$. *Thus*

$$\lim_{n\to\infty} f(a_n) = 1$$

and

$$\lim_{n\to\infty} f(b_n) = 0.$$

The limits are not equation, so $\lim_{n\to\infty} f(x)$ *does not exists.*

Example 3 *Let* $f : \mathbf{R} \to \mathbf{R}$ *defined by*

$$f(x) = \begin{cases} \cos\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

We want to show that $\lim_{n\to\infty} f(x)$ *does not exist. Let* $a_n = \frac{1}{n\pi}$. *Then* $\lim_{n\to\infty} a_n = 0$. *Also* $\cos\frac{1}{a_n} = \cos n\pi = (-1)^n$. *So* $\lim_{n\to\infty} \cos\frac{1}{a_n}$ *diverges and hence* $\lim_{n\to\infty} f(x)$ *does not exist.*

2 Rigorous definition of limit (can be skipped)

Definition 4 *Let $f(x)$ be a function, then*

$$\lim_{x \to c} f(x) = L$$

if for every $\varepsilon > 0$, there exists a real number δ such that when

$$0 < |x - c| < \delta,$$

we have

$$|f(x) - L| < \varepsilon.$$

Example 5 *We want to show that*

$$\lim_{x \to 1} x^2 = 1.$$

If $\varepsilon \geq 1$, let $\delta = 0.1$, then when $0 < |x - 1| < \delta$,

$$|x^2 - 1| = |x - 1||x + 1| \leq 0.1 \times 2.1 = 0.21 < \varepsilon.$$

If $\varepsilon < 1$, let $\delta = \frac{\varepsilon}{3}$. Then $\delta < 1$. When $0 < |x - 1| < \delta$,

$$|x^2 - 1| = |x - 1||x + 1| \leq \delta(2 + |x - 1|) < \frac{\varepsilon}{3} \times (2 + 1) = \varepsilon.$$