2017-18 MATH1010
Lecture 24: Partial fraction decomposition
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1 Partial Fraction Decomposition

In this section we investigate the antiderivatives of rational functions. Recall that rational functions are functions of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$.

We begin with an example that demonstrates the motivation behind this section. Consider the integral $\displaystyle\int \frac{1}{x^2-1}\,dx$. We do not have a simple formula for this (if the denominator were x^2+1, we would recognize the antiderivative as being the arctangent function). It can be solved using Trigonometric Substitution, but note how the integral is easy to evaluate once we realize:

$$\frac{1}{x^2-1} = \frac{1/2}{x-1} - \frac{1/2}{x+1}.$$

Thus

$$\begin{aligned}\int \frac{1}{x^2-1}\,dx &= \int \frac{1/2}{x-1}\,dx - \int \frac{1/2}{x+1}\,dx \\ &= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.\end{aligned}$$

This section teaches how to *decompose*

$$\frac{1}{x^2-1} \quad \text{into} \quad \frac{1/2}{x-1} - \frac{1/2}{x+1}.$$

We start with a rational function $f(x) = \frac{p(x)}{q(x)}$, where p and q do not have any common factors and the degree of p is less than the degree of q. It can be shown that any polynomial, and hence q, can be factored into a product of linear and irreducible quadratic terms. The following Key Idea states how to decompose a rational function into a sum of rational functions whose denominators are all of lower degree than q.

Key Idea 1
Partial Fraction Decomposition

Let $\dfrac{p(x)}{q(x)}$ be a rational function, where the degree of p is less than the degree of q.

1. **Linear Terms:** Let $(x-a)$ divide $q(x)$, where $(x-a)^n$ is the highest power of $(x-a)$ that divides $q(x)$. Then the decomposition of $\frac{p(x)}{q(x)}$ will contain the sum

$$\frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n}.$$

2. **Quadratic Terms:** Let x^2+bx+c divide $q(x)$, where $(x^2+bx+c)^n$ is the highest power of x^2+bx+c that divides $q(x)$. Then the decomposition of $\frac{p(x)}{q(x)}$ will contain the sum

$$\frac{B_1x+C_1}{x^2+bx+c} + \frac{B_2x+C_2}{(x^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(x^2+bx+c)^n}.$$

To find the coefficients A_i, B_i and C_i:

1. Multiply all fractions by $q(x)$, clearing the denominators. Collect like terms.
2. Equate the resulting coefficients of the powers of x and solve the resulting system of linear equations.

Key Idea 2
The division algorithm
If degree of p is greater or equal then q, then by division, we have

$$p(x) = q(x)d(x) + r(x),$$

where the degree of $r(x)$ is strictly smaller than the degree of $p(x)$. So

$$\frac{p(x)}{q(x)} = d(x) + \frac{r(x)}{q(x)}.$$

Example 1.1. *Decompose* $f(x) = \dfrac{1}{(x+5)(x-2)^3(x^2+x+2)(x^2+x+7)^2}$ *without solving for the resulting coefficients.* ■

Answer. The denominator is already factored, as both x^2+x+2 and x^2+x+7 cannot be factored further. We need to decompose $f(x)$

properly. Since $(x+5)$ is a linear term that divides the denominator, there will be a

$$\frac{A}{x+5}$$

term in the decomposition.

As $(x-2)^3$ divides the denominator, we will have the following terms in the decomposition:

$$\frac{B}{x-2}, \quad \frac{C}{(x-2)^2} \quad \text{and} \quad \frac{D}{(x-2)^3}.$$

The x^2+x+2 term in the denominator results in a $\dfrac{Ex+F}{x^2+x+2}$ term.

Finally, the $(x^2+x+7)^2$ term results in the terms

$$\frac{Gx+H}{x^2+x+7} \quad \text{and} \quad \frac{Ix+J}{(x^2+x+7)^2}.$$

All together, we have

$$\frac{1}{(x+5)(x-2)^3(x^2+x+2)(x^2+x+7)^2} = \frac{A}{x+5} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3} + \frac{Ex+F}{x^2+x+2} + \frac{Gx+H}{x^2+x+7} + \frac{Ix+J}{(x^2+x+7)^2}$$

Solving for the coefficients A, $B \dots J$ would be a bit tedious but not "hard."

We can also solve the variables by computer:

go to wolframalpha.com

Type `partial fraction 1/((x+5)(x-2)^3(x^2+x+2)(x^2+x+7)^2)`

Here is the answer

$$-\frac{1}{5501034(x+5)} + \frac{665617}{5015768576(x-2)} - \frac{1119}{6889792(x-2)^2} + \frac{1}{9464(x-2)^3}$$

$$\frac{-37x-39}{140800\,(x^2+x+2)} + \frac{67804x+21113}{520524225\,(x^2+x+7)} + \frac{89x-32}{296595\,(x^2+x+7)^2}$$

Example 1.2. *Rewrite* $\frac{x^5-4x^4+x^3-2x^2+x+5}{x^2-3x+1}$ *by long division as in the Key Idea 2.* ■

Answer.

$$
\begin{array}{r@{}rrrrrr}
 & x^3 & & -x^2 & -3x & -10 & \\
\cline{2-7}
x^2-3x+1 \big) & x^5-4x^4 & +x^3 & -2x^2 & +x & +5 & \\
 & -x^5+3x^4 & -x^3 & & & & \\
\cline{2-3}
 & -x^4 & & -2x^2 & & & \\
 & x^4-3x^3 & & +x^2 & & & \\
\cline{2-4}
 & & -3x^3 & -x^2 & +x & & \\
 & & 3x^3 & -9x^2 & +3x & & \\
\cline{3-5}
 & & & -10x^2 & +4x & +5 & \\
 & & & 10x^2 & -30x & +10 & \\
\cline{4-6}
 & & & & -26x & +15 &
\end{array}
$$

So

$$\frac{x^5-4x^4+x^3-2x^2+x+5}{x^2-3x+1} = x^3-2x^2-3x-10+\frac{-26x+15}{x^2-3x+1}.$$

Example 1.3. *Perform the partial fraction decomposition of* $\dfrac{1}{x^2-1}$ *and compute* $\displaystyle\int \frac{dx}{x^2-1}$. ■

Answer. The denominator factors into two linear terms: $x^2-1 = (x-1)(x+1)$. Thus

$$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}.$$

To solve for A and B, first multiply through by $x^2-1 = (x-1)(x+1)$:

$$
\begin{aligned}
1 &= \frac{A(x-1)(x+1)}{x-1} + \frac{B(x-1)(x+1)}{x+1} \\
&= A(x+1) + B(x-1) \\
&= Ax + A + Bx - B
\end{aligned}
$$

Now collect like terms.

$$= (A+B)x + (A-B).$$

The next step is key. Note the equality we have:

$$1 = (A+B)x + (A-B).$$

For clarity's sake, rewrite the left hand side as

$$0x + 1 = (A + B)x + (A - B).$$

On the left, the coefficient of the x term is 0; on the right, it is $(A + B)$. Since both sides are equal, we must have that $0 = A + B$.

Likewise, on the left, we have a constant term of 1; on the right, the constant term is $(A - B)$. Therefore we have $1 = A - B$.

We have two linear equations with two unknowns. This one is easy to solve by hand, leading to

$$\begin{array}{c} A + B = 0 \\ A - B = 1 \end{array} \Rightarrow \begin{array}{c} A = 1/2 \\ B = -1/2 \end{array}.$$

Thus

$$\frac{1}{x^2 - 1} = \frac{1/2}{x - 1} - \frac{1/2}{x + 1}.$$

Then

$$\int \frac{dx}{x^2 - 1} = \frac{1}{2} \int \frac{dx}{x - 1} - \frac{1}{2} \frac{dx}{x + 1} \quad = \frac{1}{2} \ln |x - 1| - \frac{1}{2} \log |x + 1| + C$$

A faster method for solving A and B

The denominator factors into two linear terms: $x^2 - 1 = (x - 1)(x + 1)$. Thus

$$\frac{1}{x^2 - 1} = \frac{A}{x - 1} + \frac{B}{x + 1}.$$

To solve for A and B, first multiply through by $x^2 - 1 = (x - 1)(x + 1)$:

$$\begin{aligned} 1 &= \frac{A(x - 1)(x + 1)}{x - 1} + \frac{B(x - 1)(x + 1)}{x + 1} \\ &= A(x + 1) + B(x - 1) \end{aligned}$$

Substitute $x = 1$ into the equation:

$$\begin{aligned} 1 &= 2A \\ A &= \frac{1}{2}. \end{aligned}$$

Substitute $x = -1$ into the question:

$$1 = -2B$$

$$B = -\frac{1}{2}.$$

Example 1.4. *Use partial fraction decomposition to integrate*

$$\int \frac{1}{(x-1)(x+2)^2}\ dx.$$

■

Answer. We decompose the integrand as follows, as described by Key Idea 1:

$$\frac{1}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}.$$

To solve for A, B and C, we multiply both sides by $(x-1)(x+2)^2$ and collect like terms:

$$\begin{aligned} 1 &= A(x+2)^2 + B(x-1)(x+2) + C(x-1) \qquad (1)\\ &= Ax^2 + 4Ax + 4A + Bx^2 + Bx - 2B + Cx - C \\ &= (A+B)x^2 + (4A+B+C)x + (4A-2B-C) \end{aligned}$$

Note: Equation 1 offers a direct route to finding the values of A, B and C. Since the equation holds for all values of x, it holds in particular when $x = 1$. However, when $x = 1$, the right hand side simplifies to $A(1+2)^2 = 9A$. Since the left hand side is still 1, we have $1 = 9A$. Hence $A = 1/9$.

Likewise, the equality holds when $x = -2$; this leads to the equation $1 = -3C$. Thus $C = -1/3$.

Knowing A and C, we can find the value of B by choosing yet another value of x, such as $x = 0$, and solving for B. We have

$$0x^2 + 0x + 1 = (A+B)x^2 + (4A+B+C)x + (4A-2B-C)$$

leading to the equations

$$A + B = 0, \quad 4A + B + C = 0 \quad \text{and} \quad 4A - 2B - C = 1.$$

These three equations of three unknowns lead to a unique solution:

$$A = 1/9, \quad B = -1/9 \quad \text{and} \quad C = -1/3.$$

Thus

$$\int \frac{1}{(x-1)(x+2)^2}\,dx = \int \frac{1/9}{x-1}\,dx + \int \frac{-1/9}{x+2}\,dx + \int \frac{-1/3}{(x+2)^2}\,dx.$$

Each can be integrated with a simple substitution with $u = x-1$ or $u = x+2$. The end result is

$$\int \frac{1}{(x-1)(x+2)^2}\,dx = \frac{1}{9}\ln|x-1| - \frac{1}{9}\ln|x+2| + \frac{1}{3(x+2)} + C.$$

Example 1.5. *Use partial fraction decomposition to integrate*

$$\int \frac{x^3}{(x-5)(x+3)}\,dx.$$

■

Answer. By long division

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad x \quad +2 \\ \hline x^2-2x-15 & x^3 \\ & -x^3+2x^2+15x \\ \cline{2-2} & \quad 2x^2+15x \\ & \quad -2x^2 \quad +4x+30 \\ \cline{2-2} & \quad\quad\quad 19x+30 \end{array}$$

Therefore

$$\frac{x^3}{(x-5)(x+3)} = x+2+\frac{19x+30}{(x-5)(x+3)}.$$

Using Key Idea 1, we can rewrite the new rational function as:

$$\frac{19x+30}{(x-5)(x+3)} = \frac{A}{x-5} + \frac{B}{x+3}$$

for appropriate values of A and B. Clearing denominators, we have

$$\begin{aligned} 19x+30 &= A(x+3)+B(x-5) \\ &= (A+B)x+(3A-5B). \end{aligned}$$

This implies that:

$$\begin{aligned}19 &= A + B\\ 30 &= 3A - 5B.\end{aligned}$$

Solving this system of linear equations gives

$$\begin{aligned}125/8 &= A\\ 27/8 &= B.\end{aligned}$$

Alternate method for finding A and B:

$$19x + 30 = A(x+3) + B(x-5).$$

Substitute $x = 5$

$$\begin{aligned}125 &= 8A\\ A &= \frac{125}{8}.\end{aligned}$$

Substitute $x = -3$

$$\begin{aligned}-27 &= -8B\\ B &= \frac{27}{8}.\end{aligned}$$

We can now integrate.

$$\begin{aligned}\int \frac{x^3}{(x-5)(x+3)}\ dx &= \int \left(x + 2 + \frac{125/8}{x-5} + \frac{27/8}{x+3}\right)\ dx\\ &= \frac{x^2}{2} + 2x + \frac{125}{8}\ln|x-5| + \frac{27}{8}\ln|x+3| + C.\end{aligned}$$

Example 1.6. *Use partial fraction decomposition to evaluate*

$$\int \frac{7x^2 + 31x + 54}{(x+1)(x^2+6x+11)}\ dx.$$

■

Answer. The degree of the numerator is less than the degree of the denominator so we begin by applying Key Idea 1. We have:

$$\frac{7x^2 + 31x + 54}{(x+1)(x^2+6x+11)} = \frac{A}{x+1} + \frac{Bx + C}{x^2 + 6x + 11}.$$

Now clear the denominators.

$$\begin{aligned}7x^2 + 31x + 54 &= A(x^2 + 6x + 11) + (Bx + C)(x + 1) \\ &= (A + B)x^2 + (6A + B + C)x + (11A + C).\end{aligned}$$

This implies that:

$$\begin{aligned}7 &= A + B \\ 31 &= 6A + B + C \\ 54 &= 11A + C.\end{aligned}$$

Solving this system of linear equations gives the nice result of $A = 5$, $B = 2$ and $C = -1$.

Alternate method for finding A, B and C

$$7x^2 + 31x + 54 = A(x^2 + 6x + 11) + (Bx + C)(x + 1).$$

Substitute $x = -1$, then

$$30 = 6A$$

$$A = 5.$$

There is no direct way to find B and C. Substitute $x = 0$.

$$54 = 11A + C.$$

So

$$C = -1.$$

Finally substitute $x = 1$

$$92 = 18A + 2(B - 1)$$

$$B = 2.$$

Thus

$$\int \frac{7x^2 + 31x + 54}{(x + 1)(x^2 + 6x + 11)}\, dx = \int \left(\frac{5}{x + 1} + \frac{2x - 1}{x^2 + 6x + 11}\right)\, dx.$$

The first term of this new integrand is easy to evaluate; it leads to a $5 \ln |x + 1|$ term. The second term is not hard, but takes several steps and uses substitution techniques.

The integrand $\frac{2x-1}{x^2+6x+11}$ has a quadratic in the denominator and a linear term in the numerator. This leads us to try substitution. Let $u = x^2+6x+11$, so $du = (2x+6)\ dx$. The numerator is $2x-1$, not $2x+6$, but we can get a $2x+6$ term in the numerator by adding 0 in the form of "$7-7$."

$$\begin{aligned}\frac{2x-1}{x^2+6x+11} &= \frac{2x-1+7-7}{x^2+6x+11} \\ &= \frac{2x+6}{x^2+6x+11} - \frac{7}{x^2+6x+11}.\end{aligned}$$

We can now integrate the first term with substitution, leading to a $\ln|x^2+6x+11|$ term. The final term can be integrated using arctangent. First, complete the square in the denominator:

$$\frac{7}{x^2+6x+11} = \frac{7}{(x+3)^2+2}.$$

An antiderivative of the latter term can be found by using trigonometric substitution:

$$\int \frac{7}{x^2+6x+11}\ dx = \frac{7}{\sqrt{2}}\tan^{-1}\left(\frac{x+3}{\sqrt{2}}\right) + C.$$

Let's start at the beginning and put all of the steps together.

$$\begin{aligned}\int \frac{7x^2+31x+54}{(x+1)(x^2+6x+11)}\ dx &= \int \left(\frac{5}{x+1} + \frac{2x-1}{x^2+6x+11}\right)\ dx \\ &= \int \frac{5}{x+1}\ dx + \int \frac{2x+6}{x^2+6x+11}\ dx - \int \frac{7}{x^2+6x+11}\ dx \\ &= 5\ln|x+1| + \ln|x^2+6x+11| - \frac{7}{\sqrt{2}}\tan^{-1}\left(\frac{x+3}{\sqrt{2}}\right) + C.\end{aligned}$$

Example 1.7. *Compute*

$$\int \frac{(x+2)dx}{x^4-1}.$$

■

Answer.

$$x^4-1 = (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1).$$

By partial fraction decomposition

$$\frac{x+2}{x^4-1} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}.$$

Multiplying both sides by x^4-1,

$x+2 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x-1)(x+1).$

Substitute $x = 1$

$$3 = 4A$$
$$A = \frac{3}{4}.$$

Substitute $x = -1$

$$1 = -4B$$
$$B = -\frac{1}{4}.$$

Substitute $x = 0$

$$2 = A - B - D$$
$$D = A - B - 2 = \frac{3}{4} - \frac{1}{4} - 2 = -1.$$

Substitute $x = 2$

$$4 = 15A + 5B + 6C + 3D$$
$$C = \frac{4 - 15A - 5B - 3D}{6} = -\frac{1}{2}.$$

So

$$\frac{x+2}{x^4-1} = \frac{3}{4(x-1)} - \frac{1}{4(x+1)} + \frac{-x-2}{2\,(x^2+1)}$$

$$\begin{aligned}\int \frac{-x-2}{2(x^2+1)}dx &= -\frac{1}{4}\int \frac{d(x^2+1)}{x^2+1} - \int \frac{dx}{x^2+1} \\ &= -\frac{1}{4}\ln(x^2+1) - \tan^{-1} x.\end{aligned}$$

Thus

$$\begin{aligned}\int \frac{(x+2)dx}{x^4-1} &= \int \frac{3dx}{4(x-1)} - \int \frac{dx}{4(x+1)} + \int \frac{(-x-2)dx}{2\,(x^2+1)} \\ &= \frac{3}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{4}\ln(x^2+1) - \tan^{-1} x + C\end{aligned}$$

2 Integration of $\frac{1}{(1+x^2)^n}$

In this section we will derive a reduction formula for $\displaystyle\int \frac{dx}{(1+x^2)^n}$
Let $dv = dx$, $u = \frac{1}{(1+x^2)^n}$.
Then $v = x$,

$$du = -\frac{2nxdx}{(1+x^2)^{n+1}}.$$

Hence

$$\begin{aligned}
\int \frac{dx}{(1+x^2)^n} &= \frac{x}{(1+x^2)^n} + 2n \int \frac{x^2dx}{(1+x^2)^{n+1}} \\
&= \frac{x}{(1+x^2)^n} + 2n \int \frac{(x^2+1-1)dx}{(1+x^2)^{n+1}} \\
&= \frac{x}{(1+x^2)^n} + 2n \int \frac{dx}{(1+x^2)^n} - 2n \int \frac{dx}{(1+x^2)^{n+1}}
\end{aligned}$$

Therefore

$$\int \frac{dx}{(1+x^2)^{n+1}} = \frac{1}{2n} \frac{x}{(1+x^2)^n} + \frac{2n-1}{2n} \int \frac{dx}{(1+x^2)^n}.$$

Replace n by $n-1$

$$\int \frac{dx}{(1+x^2)^n} = \frac{1}{2(n-1)} \frac{x}{(1+x^2)^{n-1}} + \frac{2n-3}{2n-2} \int \frac{dx}{(1+x^2)^{n-1}}.$$

So we have

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + C.$$

$$\begin{aligned}
\int \frac{dx}{(1+x^2)^2} &= \frac{1}{2} \frac{x}{(1+x^2)} + \frac{1}{2} \int \frac{dx}{1+x^2} \\
&= \frac{x}{2\left(x^2+1\right)} + \frac{1}{2} \tan^{-1}(x) + C.
\end{aligned}$$

$$\begin{aligned}
\int \frac{dx}{(1+x^2)^3} &= \frac{1}{4} \frac{x}{(1+x^2)^2} + \frac{3}{4} \int \frac{dx}{(1+x^2)^2} \\
&= \frac{5x}{8\left(x^2+1\right)^2} + \frac{3x^3}{8\left(x^2+1\right)^2} + \frac{3}{8} \tan^{-1}(x) + C
\end{aligned}$$

Example 2.1. *Evaluate*

$$\int \frac{dx}{(x+1)\left(x^2+4x+5\right)^2}.$$

■

Answer. By partial fraction decomposition

$$\frac{1}{(x+1)\left(x^2+4x+5\right)^2} = \frac{-x-3}{4\left(x^2+4x+5\right)} + \frac{-x-3}{2\left(x^2+4x+5\right)^2} + \frac{1}{4(x+1)}.$$

Next

$$x^2+4x+5 = (x+2)^2+1.$$

Let $u = x+2$

$$\frac{-x-3}{4\left(x^2+4x+5\right)} = \frac{-u-1}{4(u^2+1)}.$$

Hence

$$\begin{aligned}
\int \frac{(-x-3)dx}{4\left(x^2+4x+5\right)} &= -\frac{1}{4}\int \frac{udu}{u^2+1} - \frac{1}{4}\int \frac{du}{u^2+1} \\
&= -\frac{1}{8}\int \frac{d(u^2+1)}{u^2+1} - \frac{1}{4}\int \frac{du}{u^2+1} \\
&= -\frac{1}{8}\ln(u^2+1) - \frac{1}{4}\tan^{-1} u \\
&= -\frac{1}{8}\log\left(x^2+4x+5\right) - \frac{1}{4}\tan^{-1}(x+2)
\end{aligned}$$

(Let's ignore the constant for now.) Again let $u = x+2$,

$$\frac{-x-3}{2\left(x^2+4x+5\right)^2} = \frac{-u-1}{2(u^2+1)^2}.$$

Hence

$$\begin{aligned}
\int \frac{-x-3}{2\left(x^2+4x+5\right)^2} &= -\frac{1}{4}\int \frac{d(u^2+1)}{(u^2+1)^2} - \frac{1}{2}\int \frac{du}{(u^2+1)^2} \\
&= \frac{1}{4(u^2+1)} - \frac{u}{4\left(u^2+1\right)} - \frac{1}{4}\tan^{-1}(u) \\
&= -\frac{x+1}{4(x^2+4x+5)} - \frac{1}{4}\tan^{-1}(x+2)
\end{aligned}$$

So

$$\int \frac{dx}{(x+1)\left(x^2+4x+5\right)^2} = -\frac{x+1}{4\left(x^2+4x+5\right)} - \frac{1}{8}\log\left(x^2+4x+5\right)$$
$$-\frac{1}{2}\tan^{-1}(x+2) + \frac{1}{4}\log(x+1) + C$$