2017-18 MATH1010J
Lecture 19: Fundamental theorem of Calculus
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1 Inequalities of indefinite integral

Proposition 1.1. *Suppose $f(x) \leq g(x)$ on $[a, b]$, then*

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

■

Corollary 1.1.

$$\int_a^b f(x)dx \leq \int_a^b |f(x)|dx.$$

■

Proof. Let $g(x) = |f(x)|$ in the proposition. □

Corollary 1.2.

$$\left|\int f(x)dx\right| \leq \int |f(x)|dx.$$

■

Proof.

$$-|f(x)| \leq f(x) \leq |f(x)|.$$

So

$$-\int_a^b |f(x)|dx \leq \int_a^b f(x)dx \leq \int_a^b |f(x)|dx.$$

The result follows. □

Corollary 1.3. *Let M (resp. m) be the maximum (resp. minimum) value of $f(x)$ on $[a, b]$ Then*

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

■

Proof. For $x \in [a, b]$, $m \leq f(x) \leq M$. Hence

$$\int_a^b m dx \leq \int_a^b f(x) dx \leq \int_a^b M dx.$$

□

Proposition 1.2 (Mean value theorem for definite integral)**.** *Suppose $f(x)$ is a continuous function on $[a, b]$. Then there exists $c \in [a, b]$ such that*

$$f(c) = \frac{1}{b-a} \int_a^b f(x) dx.$$

■

Proof. By the previous corollary (use the same notation)

$$m \leq \frac{1}{b-a} \int_a^b f(x) dx \leq M.$$

Suppose $f(x_1) = m$ and $f(x_2) = M$, $x_1, x_2 \in [a, b]$. By the intermediate value theorem, there exists c between x_1 and x_2 (hence in $[a, b]$) such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x).$$

□

2 Fundamental theorem of Calculus

Theorem 2.1 (The Fundamental Theorem of Calculus)**.** *If the function $f(x)$ is continuous on the interval $a \leq x \leq b$, then*

$$\int_a^b f(x)\, dx = F(b) - F(a) \tag{1}$$

where $F(x)$ is any antiderivative of $f(x)$ on $a \leq x \leq b$. ■

Proof. The proof will be given later. □

Example 2.1. *Evaluate* $\int_1^2 x\, dx$.

Answer. The function $F(x) = \frac{1}{2}x^2$ is an antiderivative of $f(x) = x$; thus, from (1)

$$\int_1^2 x\,dx = \frac{1}{2}x^2\Big|_1^2 = \frac{1}{2}(2)^2 - \frac{1}{2}(1)^2 = 2 - \frac{1}{2} = \frac{3}{2}.$$

■

The Relationship between Definite and Indefinite Integrals
Let F be any antiderivative of the integrand on $[a, b]$, and let C be any constant; then

$$\int_a^b f(x)\,dx = [f(x) + C]_a^b = [F(b) + C] - [F(a) + C] = F(b) - F(a).$$

Thus, for purpose of evaluating a definite integral we can omit the constraint of integration in

$$\int_a^b f(x)\,dx = [F(x) + C]_a^b$$

and express (1) as

$$\int_a^b f(x)\,dx = \int f(x)\,dx\Big|_a^b.$$

Example 2.2. *Compute*

$$\int_1^9 \sqrt{x}\,dx.$$

Answer.

$$\int_1^9 \sqrt{x}\,dx = \int x^{1/2}\,dx\Big|_1^9 = \frac{2}{3}x^{3/2}\Big|_1^9 = \frac{2}{3}(27 - 1) = \frac{52}{3}.$$

■

3 Fundamental theorem of Calculus (another form)

Theorem 3.1. *Suppose $f(x)$ is a continuous function on $[a, b]$ and $x \in [a, b]$. Let*

$$F(x) = \int_a^x f(t)dt.$$

Then $F(x)$ is the anti-derivative of $f(x)$, i.e.

$$F'(x) = f(x).$$

■

Proof. By Proposition 1.2

$$\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(x) = f(c_h)$$

for some c_h between x and $x+h$, then $h \to 0$, $c_h \to x$. Therefore

$$\lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = f(x).$$

□

Proof of the fundamental theorem of calculus. By the previous theorem $F(x) = \int_a^x f(t)dt$ is the antiderivative of $f(x)$ and $F(a) = 0$. Then

$$\int_a^b f(x)dt = F(b) = F(b) - F(a).$$

□

Example 3.1. *Compute the following*

1. $\dfrac{d}{dx}\displaystyle\int_2^x \sin(t)dx.$
2. $\dfrac{d}{dx}\displaystyle\int_x^3 e^{-t^3}dt.$
3. $\dfrac{d}{dx}\displaystyle\int_0^{x^3} \sqrt{2+\sin t}dt.$
4. $\dfrac{d}{dx}\displaystyle\int_{x^2}^{x^3} \frac{dt}{\ln t}$ *for* $x > 0.$

Answer.

1. By theorem 3.1, $\dfrac{d}{dx}\displaystyle\int_2^x \sin(t)dx = \sin(x).$

2. $\displaystyle\int_x^3 e^{-t^3}dt = -\int_3^x e^{-t^3}dt$. By theorem 3.1,

$$\frac{d}{dt}\int_x^3 e^{-t^3}dt = -\frac{d}{dt}\int_3^x e^{-t^3}dt = -e^{-x^3}.$$

3. We can use the chain rule. Let $u = x^3$, $y = \displaystyle\int_0^{x^3} \sqrt{2+\sin t}dt = \int_1^u \sqrt{2+\sin t}dt$.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dt} = \sqrt{2+\sin u}(3x^2) = 3x^2\sqrt{2+\sin(x^3)}.$$

4. Let $a = 1$.

$$\int_{x^2}^{x^3} \frac{1}{\ln t}dt = \int_a^{x^3} \frac{dt}{\ln t}dt - \int_a^{x^2} \frac{dt}{\ln t}dt.$$

Let $u = x^3$, $y = \displaystyle\int_a^{x^3} \frac{dt}{\ln t}dt = \int_a^u \frac{dt}{\ln t}$.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{\ln u}(3x^2) = \frac{x^2}{\ln x}.$$

Similarly let $u = x^2$, $y = \displaystyle\int_a^{x^2} \frac{dt}{\ln t}$.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dt} = \frac{1}{\ln u}(2x) = \frac{x}{\ln x}.$$

Therefore

$$\frac{d}{dx}\int_{x^2}^{x^3} \frac{dt}{\ln t}dt = \frac{x^2}{\ln x} - \frac{x}{\ln x}.$$

■

Generally, let $u(x)$, $v(x)$ be differentiable function and $f(x)$ a continuous function, then

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = f(v(x))v'(x) - f(u(x))u'(x).$$

Let c be a constant

$$\int_{u(x)}^{v(x)} f(t)dt = \int_c^{v(x)} f(t)dt - \int_c^{u(x)} f(t)dt.$$

Let $F(v) = \int_c^v f(t)dt$. Then $F'(v) = f(v)$. Let $v = v(x)$, by the chain rule

$$\frac{d}{dx}\int_c^{v(x)} f(t)dt = \frac{d}{dx}F(v(x)) = F'(v(x))v'(x).$$

Similarly Let $G(u) = \int_c^u f(t)dt$. Then $G'(u) = f(u)$. Let $u = u(x)$, by the chain rule

$$\frac{d}{dx}\int_c^{u(x)} f(t)dt = \frac{d}{dx}F(u(x)) = F'(u(x))u'(x).$$

Remark: Don't use the above formula directly in the tests or exam because you have show your steps. Follow the above procedure and write down your steps clearly.

4 Definite integral of piece functions

Example 4.1. *Evaluate* $\int_0^3 f(x)\,dx$ *if*

$$f(x) = \begin{cases} x^2, & x < 2 \\ 3x - 2, & x \geq 2 \end{cases}$$

Answer.

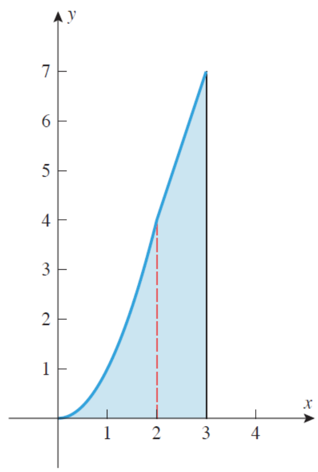
We can integrate from 0 to 2 and from 2 to 3 separately and add the results. This yields

$$\begin{aligned} \int_0^3 f(x)\,dx &= \int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx = \int_0^2 x^2\,dx + \int_2^3 (3x-2)\,dx \\ &= \left.\frac{x^3}{3}\right|_0^2 + \left[\frac{3x^2}{2} - 2x\right]_2^3 = \left(\frac{8}{3} - 0\right) + \left(\frac{15}{2} - 2\right) = \frac{49}{6}. \end{aligned}$$

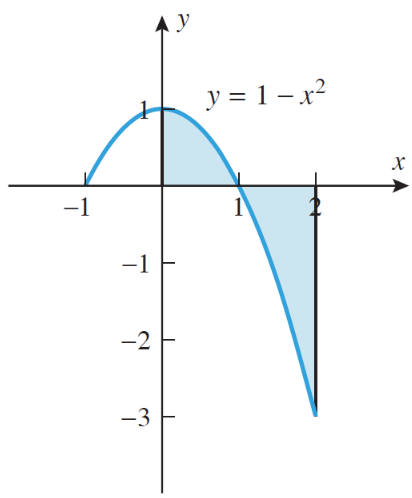
■

If f is a continuous function on the interval $[a, b]$, then we define the *total area* between the curve $y = f(x)$ and the interval $[a, b]$ to be

$$\text{total area} = \int_a^b |f(x)|\,dx.$$



Example 4.2. *Find the total area between the curve* $y = 1 - x^2$ *and the* x*-axis over the interval* $[0, 2]$.



Answer. The are is given by

$$\begin{aligned}\int_0^2 |1 - x^2|\, dx &= \int_0^1 (1 - x^2) dx + \int_1^2 -(1 - x^2) dx \\ &= \left[x - \frac{x^3}{3}\right]_0^1 - \left[x - \frac{x^3}{3}\right]_1^2 \\ &= \frac{2}{3} - \left(-\frac{4}{3}\right) = 2.\end{aligned}$$

■

Example 4.3. *Let* $f(x) = x(x-1)(x-2)$. *Compute*

$$\int_0^4 |f(x)|dx.$$

Answer. For $0 \leq x \leq 1$, $f(x) \geq 0$. So $|f(x)| = f(x)$.
For $1 \leq x \leq 2$, $f(x) \leq 0$. So $|f(x)| = -f(x)$.
For $x \geq 2$, $f(x) \geq 0$. So $|f(x)| = f(x)$.
Therefore

$$\int_0^4 |f(x)|dx = \int_0^1 f(x)dx + \int_1^2 (-f(x))dx + \int_2^4 f(x)dx.$$

$$= \left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 - \left[\frac{x^4}{4} - x^3 + x^2\right]_1^2 + \left[\frac{x^4}{4} - x^3 + x^2\right]_2^4 = \frac{33}{2}.$$

■

5 Area between curves

I suppose you have already learned this in the secondary school. Should be skipped.

Theorem 5.1. *Let* $f(x)$ *and* $g(x)$ *be continuous functions defined on* $[a, b]$ *where* $f(x) \geq g(x)$ *for all* x *in* $[a, b]$. *The area of the region bounded by the curves* $y = f(x)$, $y = g(x)$ *and the lines* $x = a$ *and* $x = b$ *is*

$$\int_a^b \big(f(x) - g(x)\big)\ dx.$$

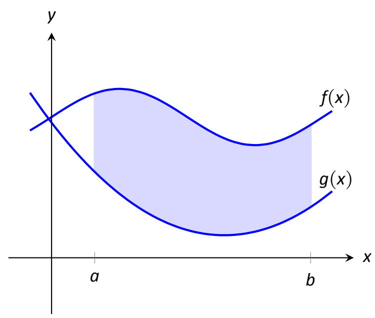
■

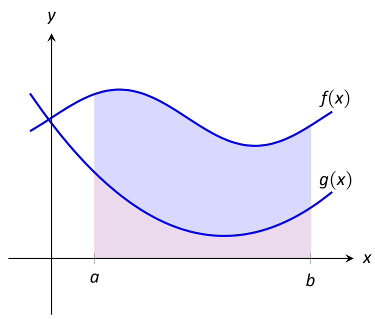
Proof. The area between $f(x)$ and $g(x)$ is obtained by subtracting the area under g from the area under f. Thus the area is

$$\int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b (f(x) - g(x))dx.$$

□

Example 5.1. *Find the area of the region enclosed by* $y = x^2 + x - 5$ *and* $y = 3x - 2$.



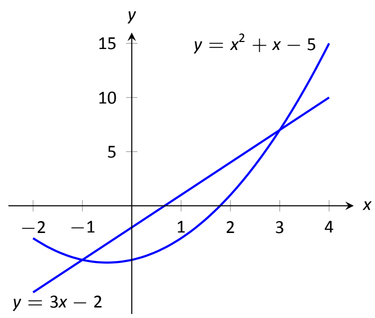


Answer. The region whose area we seek is completely bounded by these two functions; they seem to intersect at $x = -1$ and $x = 3$. To check, set $x^2 + x - 5 = 3x - 2$ and solve for x:

$$\begin{aligned} x^2 + x - 5 &= 3x - 2 \\ (x^2 + x - 5) - (3x - 2) &= 0 \\ x^2 - 2x - 3 &= 0 \\ (x-3)(x+1) &= 0 \\ x &= -1,\ 3. \end{aligned}$$

The area is

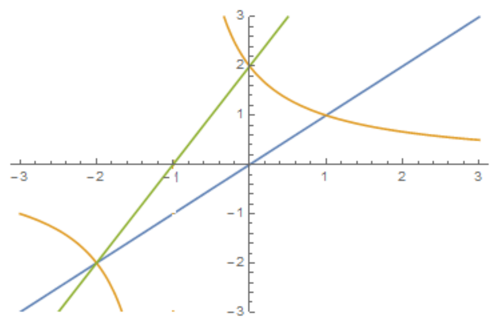
$$\begin{aligned} \int_{-1}^{3} \big(3x - 2 - (x^2 + x - 5)\big)\ dx &= \int_{-1}^{3} (-x^2 + 2x + 3)\ dx \\ &= \left(-\frac{1}{3}x^3 + x^2 + 3x\right)\Bigg|_{-1}^{3} \\ &= -\frac{1}{3}(27) + 9 + 9 - \left(\frac{1}{3} + 1 - 3\right) \\ &= 10\frac{2}{3}. \end{aligned}$$



■

Example 5.2. *Find the area bounded by*

$$y = f(x) = x, y = g(x) = \frac{2}{x+1} \text{ and } y = h(x) = 2x + 2.$$



Answer. Area is

$$\int_{-2}^{0} (h(x) - f(x))dx + \int_{0}^{1} (g(x) - f(x))dx$$

$$= \int_{-2}^{0} (2x + 2 - x) + \int_{0}^{1} (\frac{2}{x+1} - x)dx$$

$$= \left[\frac{x^2}{2} + 2x\right]_{-2}^{0} + \left[2\ln|x+1| - \frac{x^2}{2}\right]_{0}^{1}$$

$$= 2 + (2\ln 2 - \frac{1}{2}) = \frac{3}{2} + \ln 4.$$

■