2017-18 MATH1010J
Lecture 18: The definite integral
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1 The definite integral as area

Suppose f is a function on $[a, b]$. Suppose further that $f(x)$ is positive on $[a, b]$. Then we define

$$\int_a^b f(x)dx = \text{ area between } f(x) \text{ and the } x\text{-axis.}$$

What if some of the value of $f(x)$ is negative? Because $f(x)$ is negative, the "height" of at this point is negative, so we take the area as negative. Therefore we have the following definition.

Definition 1.1 (The Definite Integral, Total Signed Area)**.** *Let $y = f(x)$ be defined on a closed interval $[a, b]$. The **total signed area from** $x = a$ **to** $x = b$ **under** f is:*

(area under f and above the x–axis on $[a, b]$) − (area above f and under the x–axis on $[a, b]$).

*The **definite integral of** f **on** $[a, b]$ is the total signed area of f on $[a, b]$, denoted*

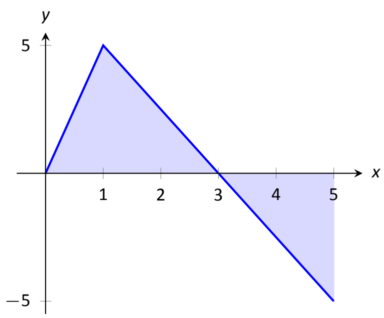
$$\int_a^b f(x)\ dx,$$

*where a and b are the **bounds of integration.*** ■

By our definition, the definite integral gives the "signed area under f." We usually drop the word "signed" when talking about the definite integral, and simply say the definite integral gives "the area under f" or, more commonly, "the area under the curve."

Example 1.1. *Consider the function f given below. Compute $\int_0^5 f(x)dx$.*

Answer. The graph of f is above the x-axis on $[0, 3]$. The area is $\frac{1}{2} \times 3 \times 1 = 1.5$.



The graph of f is under the x-axis on $[3, 5]$. This is the "negative" area. The area is $-\frac{1}{2} \times 2 \times 1 = -1$. Hence

$$\int_0^5 f(x)dx = 1.5 - 1 = 0.5.$$

■

Use the geometric interpretation, we have

Theorem 1.1 (Properties of the Definite Integra)**.** *Let f and g be defined on a closed interval I that contains the values a, b and c, and let k be a constant. The following hold:*

1. $\displaystyle\int_a^a f(x)\ dx = 0$
2. $\displaystyle\int_a^b f(x)\ dx + \int_b^c f(x)\ dx = \int_a^c f(x)\ dx$
3. $\displaystyle\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$
4. $\displaystyle\int_a^b \big(f(x) \pm g(x)\big)\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$
5. $\displaystyle\int_a^b k \cdot f(x)\ dx = k \cdot \int_a^b f(x)\ dx$
6. *Suppose* $f(x) \leq g(x)$, *then* $\displaystyle\int f(x)\ dx \leq \int g(x)\ dx.$

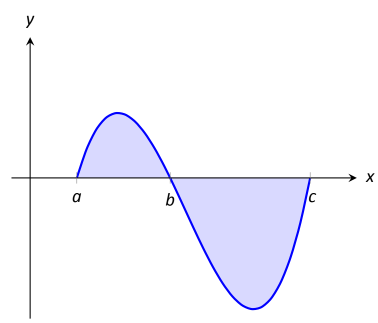
■

Proof. We give the geometric interpretation for some properties. For rigorous proof we need something call the **Riemann sum**.
For property 1, area over a single point a is 0. For property 2, area over $[a, b]$ is same as summation of areas over $[a, c]$ and $[c, b]$. What if $a < b < c$? Then we need property 3.
In fact, if property 2 is true.

$$\int_a^b f(x)\ dx + \int_b^a f(x)\ dx = \int_a^a f(x)\ dx = 0.$$

So property 3 is valid. □

Example 1.2. *Consider the graph of a function $f(x)$ shown below.*



Answer the following:

1. *Which value is greater:* $\int_a^b f(x)\ dx$ *or* $\int_b^c f(x)\ dx$*?*

2. *Is* $\int_a^c f(x)\ dx$ *greater or less than 0?*

3. *Which value is greater:* $\int_a^b f(x)\ dx$ *or* $\int_c^b f(x)\ dx$*?*

Answer.

1. $\int_a^b f(x)\ dx$ has a positive value (since the area is above the x–axis) whereas $\int_b^c f(x)\ dx$ has a negative value. Hence $\int_a^b f(x)\ dx$ is bigger.

2. $\int_a^c f(x)\ dx$ is the total signed area under f between $x = a$ and $x = c$. Since the region below the x–axis looks to be larger than the region above, we conclude that the definite integral has a value less than 0.

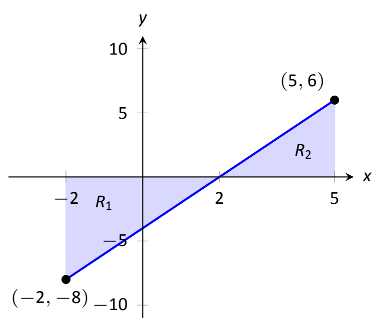
3. Note how the second integral has the bounds "reversed." Therefore $\int_c^b f(x)\ dx$ represents a positive number, greater than the area described by the first definite integral. Hence $\int_c^b f(x)\ dx$ is greater.

■

Example 1.3. *Evaluate the following definite integrals:*

$$1.\ \int_{-2}^{5} (2x-4)\ dx \qquad 2.\ \int_{-3}^{3} \sqrt{9-x^2}\ dx.$$

Answer.



1. We see we need to compute the areas of two regions, which we have labeled R_1 and R_2. Both are triangles, so the area computation is straightforward:

$$R_1 : \frac{1}{2}(4)(8) = 16 \qquad R_2 : \frac{1}{2}(3)6 = 9.$$

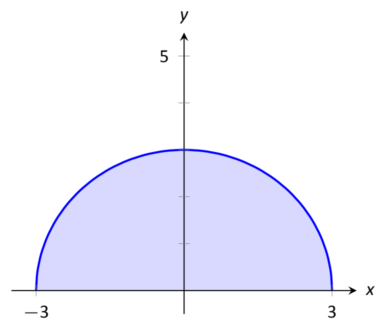
Region R_1 lies under the x–axis, hence it is counted as negative area (we can think of the triangle's height as being "−8"), so

$$\int_{-2}^{5} (2x-4)\ dx = -16 + 9 = -7.$$

2. Recognize that the integrand of this definite integral describes a half circle with radius 3. Thus the area is:

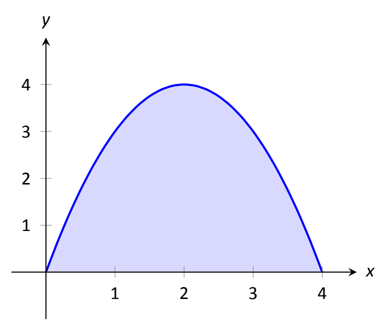
$$\int_{-3}^{3} \sqrt{9-x^2}\ dx = \frac{1}{2}\pi r^2 = \frac{9}{2}\pi.$$

■



2 Riemman Sum

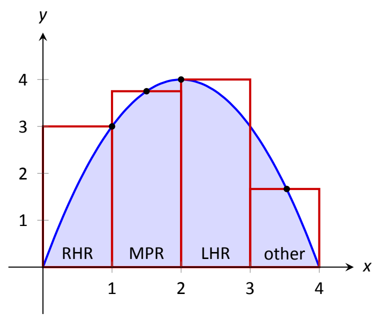
Consider the region given below, which is the area under $y = 4x - x^2$ on $[0, 4]$. What is the signed area of this region – i.e., what is $\int_0^4 (4x - x^2)\ dx$?



We can 4 rectangles of equal width of 1. This *partitions* the interval $[0, 4]$ into 4 *subintervals*, $[0, 1]$, $[1, 2]$, $[2, 3]$ and $[3, 4]$. On each subinterval we will draw a rectangle.

There are three common ways to determine the height of these rectangles: the **Left Hand Rule**, the **Right Hand Rule**, and the **Midpoint Rule**. The **Left Hand Rule** says to evaluate the function at the left–hand endpoint of the subinterval and make the rectangle that height. In the figure below, the rectangle drawn on the interval $[2, 3]$ has height determined by the Left Hand Rule; it has a height of $f(2)$. (The rectangle is labeled "LHR.")

The **Right Hand Rule** says the opposite: on each subinterval, evaluate the function at the right endpoint and make the rectangle that height. In the figure, the rectangle drawn on $[0, 1]$ is drawn using $f(1)$ as its height; this rectangle is labeled "RHR.".



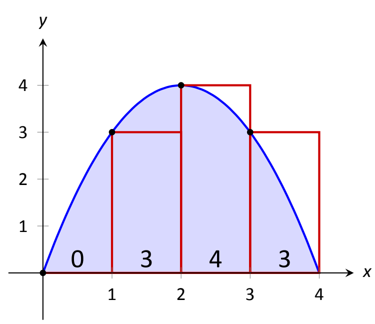
The **Midpoint Rule** says that on each subinterval, evaluate the function at the midpoint and make the rectangle that height. The rectangle drawn on $[1,2]$ was made using the Midpoint Rule, with a height of $f(1.5)$. That rectangle is labeled "MPR."

These are the three most common rules for determining the heights of approximating rectangles, but one is not forced to use one of these three methods. The rectangle on $[3,4]$ has a height of approximately $f(3.53)$, very close to the Midpoint Rule. It was chosen so that the area of the rectangle is *exactly* the area of the region under f on $[3,4]$. (Later you'll be able to figure how to do this, too.)

Exercise 2.1. *Approximate the value of* $\int_0^4 (4x - x^2)\ dx$ *using the Left Hand Rule, the Right Hand Rule, and the Midpoint Rule, using 4 equally spaced subintervals.*

Answer.

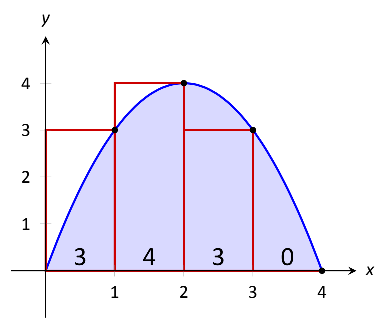
1. We break the interval $[0,4]$ into four subintervals as before. In the figure below, we see 4 rectangles drawn on $f(x) = 4x - x^2$ using the Left Hand Rule. (The areas of the rectangles are given in each figure.) We add up the areas of each rectangle



(height× width) for our Left Hand Rule approximation:

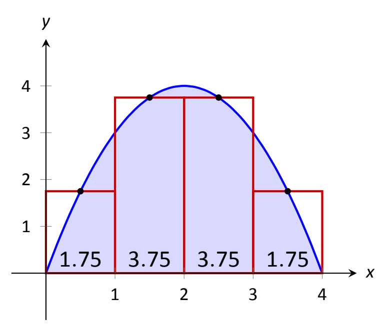
$$f(0)\cdot 1 + f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 =$$
$$0 + 3 + 4 + 3 = 10.$$

2. Right hand rule.



$$f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1 =$$
$$3 + 4 + 3 + 0 = 10.$$

3. Midpoint rule.

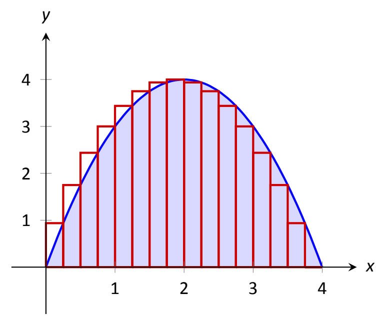


This gives an approximation of $\int_0^4 (4x - x^2)\,dx$ as:

$$f(0.5)\cdot 1 + f(1.5)\cdot 1 + f(2.5)\cdot 1 + f(3.5)\cdot 1 =$$
$$1.75 + 3.75 + 3.75 + 1.75 = 11.$$

We get values 10 or 11. ■

The above approximation is not good enough.



Example 2.1. *Approximate* $\int_0^4 (4x - x^2)\, dx$ *using the Right Hand Rule and summation formulas with 16 equally spaced intervals.*

Answer. Write $f(x) = 4x - x^2$. We divide the interval $[0, 4]$ into 16 equal intervals. Each interval has length $\Delta x = \frac{4-0}{16} = 0.25$. The points divide the interval are denoted by

$$0 = x_0 < x_1 < x_2 < \cdots < x_{15} < x_{16} = 4.$$

Because $x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \cdots = x_{16} - x_{15} = 0.25$, we have

$$x_i = 0.25i.$$

The approximation by the Right Hand Rule is

$$\sum_{i=1}^{16} f(x_i)\Delta x = \sum_{i=1}^{16} f(x_i)\Delta x$$

$$= 0.25 \times f(0.25) + 0.25 \times f(0.5) + \cdots 0.25 \times f(4) = 10.625$$

■

We now discuss the general formula for summation using Left Hand, Right Hand and Midpoint Rules to approximate $\int_a^b f(x)\, dx$ with n equally spaced intervals
We first divide $[a, b]$ into n equal intervals. Each interval has length

$$\Delta x = \frac{b-a}{n}.$$

The points divided the intervals are denoted by

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Because $x_i - x_{i-1} = \Delta x$. We have

$$x_i = a + i\Delta x_i.$$

Theorem 2.1. *The summation formula using Left Hand Rules to approximate* $\int_a^b f(x)dx$ *with* n *equally space intervals is*

$$\sum_{i=1}^{n} f(x_{i-1})\Delta x = \frac{b-a}{n}\sum_{i=1}^{n} f(a+(i-1)\frac{b-a}{n}). \tag{1}$$

Using Right Hand Rule

$$\sum_{i=1}^{n} f(x_i)\Delta x = \frac{b-a}{n}\sum_{i=1}^{n} f(a+i\frac{b-a}{n}). \tag{2}$$

Using Midpoint rule

$$\sum_{i=1}^{n} f(\frac{x_{x-1}+x_i}{2})\Delta x = \frac{b-a}{n}\sum_{i=1}^{n} f(a+(i-\frac{1}{2})\frac{b-a}{n}). \tag{3}$$

■

Example 2.2. *Approximate* $\int_0^4 (4x-x^2)\ dx$ *using the Right Hand Rule and summation formulas with* n *equally spaced intervals.*

Answer. By (2),

$$\begin{aligned}\frac{4}{n}\sum_{i=1}^{n} f(i\frac{4}{n}) &= \frac{4}{n}\sum_{i=1}^{n}\left(\frac{16i}{n}-\frac{16i^2}{n^2}\right)\\ &= \frac{64}{n^2}\sum_{i=1}^{n} - \frac{64}{n^3}\sum_{i=1}^{n} i^2\\ &= \frac{64}{n^2}\frac{n(n+1)}{2} - \frac{64}{n^3}\frac{n(n+1)(2n+1)}{6}\\ &= \frac{32}{3}\left(1-\frac{1}{n^2}\right).\end{aligned}$$

Here we use

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

■

The summations for the Left Hand, Right Hand and Midpoint Rules looked like. Each had the same basic structure, which was:

1. each rectangle has the same width, which we referred to as Δx, and

2. each rectangle's height is determined by evaluating f at a particular point in each subinterval. For instance, the Left Hand Rule states that each rectangle's height is determined by evaluating f at the left hand endpoint of the subinterval the rectangle lives on.

One could partition an interval $[a, b]$ with subintervals that did not have the same size. We refer to the length of the first subinterval as Δx_1, the length of the second subinterval as Δx_2, and so on, giving the length of the i th subinterval as Δx_i. Also, one could determine each rectangle's height by evaluating f at *any* point in the i th subinterval. We refer to the point picked in the first subinterval as c_1, the point picked in the second subinterval as c_2, and so on, with c_i representing the point picked in the i th subinterval. Thus the height of the i th subinterval would be $f(c_i)$, and the area of the i th rectangle would be $f(c_i)\Delta x_i$.

Summations of rectangles with area $f(c_i)\Delta x_i$ are named after mathematician Georg Friedrich Bernhard Riemann, as given in the following definition.

Definition 2.1 (Riemann Sum). *Let f be defined on the closed interval $[a, b]$ and let*

$$a = x_0 < x_1 < \ldots < x_n = b$$

is a partition of $[a, b]$. Let Δx_i denote the length of the i th subinterval $[x_{i-1}, x_i]$ and let c_i denote any value in the i th subinterval.

The sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i$$

*is a **Riemann sum** of f on $[a, b]$.* ■

Example 2.3. *Using the Right Hand Rule, we take*

$$\Delta x_i = \frac{b-a}{n}, \; x_i = a + i\frac{b-a}{n}, c_i = x_i.$$

For Left Hand Rule,

$$c_i = x_{i-1}.$$

For Mid Point Rule,

$$c_i = \frac{x_{i-1} + x_i}{2}.$$

Example 2.4. *Suppose f attain maximum (resp. minimum) on $[x_{i-1}, x_i]$ at $x = M_i$ (resp. $x = m_i$). The the corresponding Riemann sum is bigger (resp. smaller) than $\int_a^b f(x)dx$, i.e.*

$$\sum_{i=1}^{n} f(M_i)\Delta x_i \geq \int_a^b f(x)dx \geq \sum_{i=1}^{n} f(m_i)\Delta x_i.$$

Example 2.5. *Approximate $\int_0^1 e^x$ with $n = 10$.*
$x_0 = 0$, $x_1 = 0.01$, $x_2 = 0.04$, $x_3 = 0.09$, $x_4 = 0.16$, $x_5 = 0.25$, $x_6 = 0.36$, $x_7 = 0.49$, $x_8 = 0.64$, $x_9 = 0.81$, $x_{10} = 1$.
$c_1 = 0.001$, $c_2 = 0.011$, $c_3 = 0.041$, $c_4 = 0.091$, $c_5 = 0.161$, $c_6 = 0.251$, $c_7 = 0.361$, $c_8 = 0.491$, $c_9 = 0.641$, $c_{10} = 0.811$.
Answer. *$\Delta x_1 = x_1 - x_0 = 0.01$, $\Delta x_2 = x_2 - x_1 = 0.03$, $\Delta x_3 = x_3 - x_2 = 0.05$, $\Delta x_4 = x_4 - x_3 = 0.07$, $\Delta x_5 = x_5 - x_4 = 0.09$, $\Delta x_6 = x_6 - x_5 = 0.11$, $\Delta x_7 = x_7 - x_6 = 0.13$, $\Delta x_8 = x_8 - x_7 = 0.15$, $\Delta x_9 = x_9 - x_8 = 0.17$, $\Delta x_{10} = x_{10} - x_9 = 0.19$. The Riemann sum is*

$$\sum_{i=1}^{10} f(x_i)\Delta x_i = 0.1597$$

■

Example 2.6. *Approximate $\int_{-1}^{5}(2x-3)dx$ by the Right Hand Rule, Left Hand Rule and Middle Point Rule with n equally spaced subintervals, then take the limit as $n \to \infty$ to find the exact area.*

Answer. By (1), the Riemann sum using the Left Hand Rule is

$$\begin{aligned}\frac{6}{n}\sum_{i=1}^{n}\left(2(-1+\frac{6(i-1)}{n})-3\right) &= \frac{6}{n}\sum_{i=1}^{n}(-5-\frac{12i}{n}) \\ &= \frac{6}{n}\left(-5n-\frac{12}{n}\frac{n(n+1)}{2}\right) \\ &= 6-\frac{36}{n}.\end{aligned}$$

The area is

$$\lim_{n\to\infty} 6 - \frac{36}{n} = 6.$$

By (2), the Riemann sum using the Right hand Rule is

$$\begin{aligned}\frac{6}{n}\sum_{i=1}^{n}\left(2(-1+\frac{6i}{n})-3\right) &= \frac{6}{n}\sum_{i=1}^{n}(-5+\frac{12i}{n}) \\ &= \frac{6}{n}\left(-5n+\frac{12}{n}\frac{n(n+1)}{2}\right) \\ &= 6+\frac{36}{n}.\end{aligned}$$

The area is

$$\lim_{n\to\infty} 6+\frac{36}{n} = 6.$$

By (3), the Riemann sum using the Right Hand Rule is

$$\begin{aligned}\frac{6}{n}\sum_{i=1}^{n}\left(2(-1+\frac{6(i-1/2)}{n})-3\right) &= \frac{6}{n}\sum_{i=1}^{n}(-5+\frac{12i}{n}-\frac{6}{n}) \\ &= \frac{6}{n}\left(-5n-\frac{12}{n}\frac{n(n+1)}{2}-6\right) \\ &= 6.\end{aligned}$$

The area is

$$\lim_{n\to\infty} 6 = 6.$$

■

3 Limit of the Riemann Sum

We have used limits to evaluate exactly given definite limits. Will this always work? We will show, given not–very–restrictive conditions, that yes, it will always work.

Let $S_L(n)$ ($S_R(n)$, $S_M(n)$ resp.) be the Riemann sum using the Left Hand Rule (1) (Right Hand Rule (2) , Middle Point Rule (3) resp.).

Theorem 3.1. *If f is a continuous function on $[a,b]$, then*

$$\lim_{n\to\infty} S_L(n) = \lim_{n\to\infty} S_R(n) = \lim_{n\to\infty} S_M(n) = \int_a^b f(x)dx.$$

■

Theorem 3.2. *If f is a continuous function on $[a, b]$, then the limit of the Riemann sum*

$$\lim_{\max\{\Delta x_i\}\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

the limit is used to define $\int_a^b f(x)dx$, i.e.

$$\lim_{\max\{\Delta x_i\}\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)dx.$$

■

We can use the above theorem to prove Theorem 1.1 Property 4 (other properties can be proved similarly) the Riemman sum for $f + g$ is

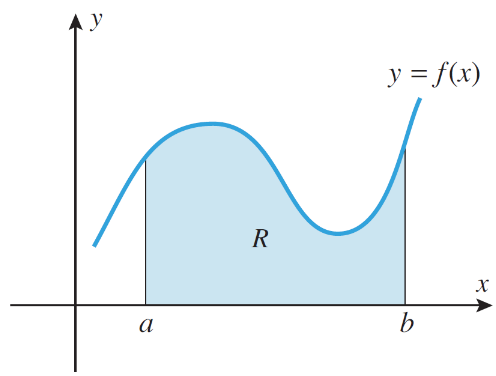
$$\sum_{i=1}^{n}(f(c_i) + g(c_i)\Delta x_i = \sum_{i=1}^{n} f(c_i)\Delta x_i + \sum_{i=1}^{n} g(c_i)\Delta x_i.$$

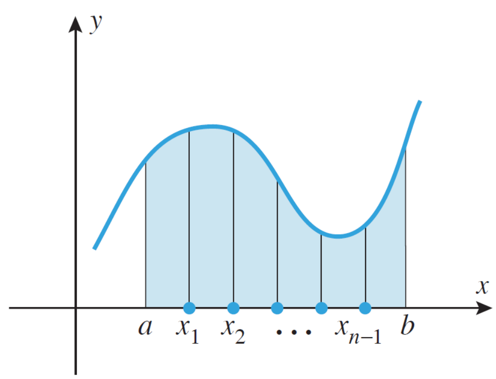
The result follows by letting $\max\{\Delta x_i\} \to 0$.

4 Appendix: Geometric interpretation

In below is the geometric interpretation

1. we want to find the area
2. Divide the interval
3. Use rectangles to approximate the area
4. Calculate the area of each rectangle and sum them up
5. More intervals give better approximation


y
y = f(x)
R
x
a
b


y
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...
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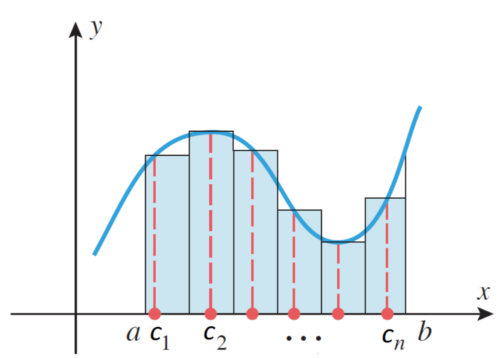
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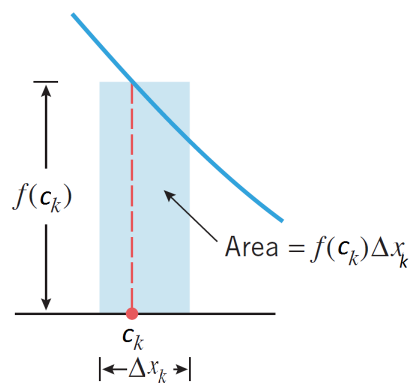
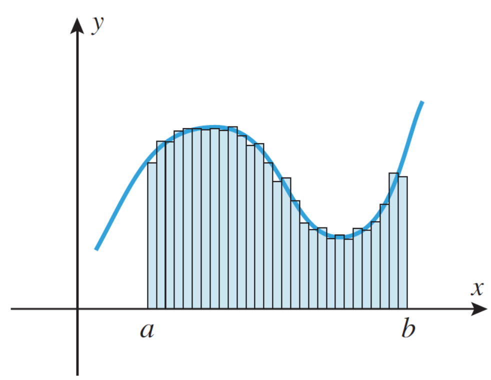


$f(c_k)$
Area = $f(c_k)\Delta x_k$
c_k
Δx_k



y
x
a
b