**2017-18 MATH1010**
**Lecture 14: The mean value theorem**
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# 1 The mean value theorems

**Theorem 1.1** (The extreme value theorem). *If $f$ is continuous on a closed interval $[a, b]$, then $f$ attains both an absolute max and absolute minimum value in $[a, b]$.* ■

The proof is difficult. So we will skip the proof.
Recall the following

**Proposition 1.1.** *If $f$ is continuous on a closed interval $[a, b]$, and differentiable on $(a, b)$. If $f$ attains maximum or minimum at $x = c$, then $f'(c) = 0$.* ■

*Proof.* Without loss of generality, we can assume $f(c)$ is the absolute maximum, so for very $h > 0$, $f(c + h) \leq f(c)$. Hence

$$f'(c) = \lim_{h\to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0.$$

Similarly for $h < 0$, $f(c + h) \leq f(c)$.

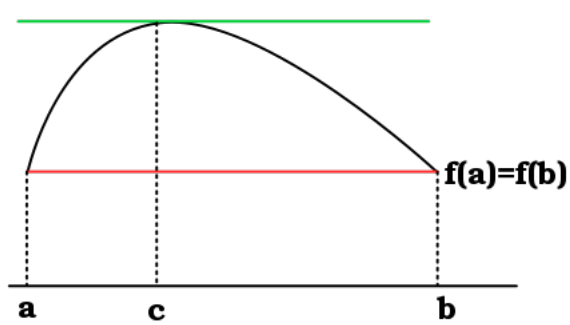
$$f'(c) = \lim_{h\to 0^-} \frac{f(c+h) - f(c)}{h} \geq 0.$$

Hence $f'(c) = 0$. □

**Theorem 1.2** (Rolle's theorem). *Suppose $f$ is a function on $[a, b]$ and satisfies the following conditions*

1. *$f(x)$ is continuous on $[a, b]$.*
2. *$f(x)$ is differentiable on $(a, b)$.*
3. *$f(a) = f(b)$.*

*Then there exists $c \in (a, b)$, such that $f'(c) = 0$.*
**Remark**: *pay attention to whether we use $[a, b]$ or $(a, b)$.* ■




*Proof.* By the extreme value theorem, $f$ attains both maximum and minimum values on $[a, b]$.

If both absolute maximum and absolute minimum attains on the end points $x = a$ and $x = b$, then the function is a constant function since $f(a) = f(b)$. So $f'(x) = 0$ for all $x \in [a, b]$. The theorem is trivial for this case. So we can at least one extreme value attains at $c \in (a, b)$. So $f'(c) = 0$. □

**Example 1.1.** *Let* $f(x) = x^4 - x^3 + 1$. *Show that there exists a number* $c \in (0, 1)$ *such that* $f'(c) = 0$. ■

**Answer.** $f(x)$ is continuous on $[0, 1]$, differentiable on $(0, 1)$. Also $f(0) = 1 = f(1)$. So there exists $c \in (0, 1)$ such that $f'(c) = 0$.
In fact

$$f'(c) = 4c^3 - 3c^2 = 0.$$

So we can take

$$c = \frac{3}{4}.$$

■

**Theorem 1.3** (Mean value theorem)**.** *Suppose* $f$ *is a function on* $[a, b]$ *and satisfies the following conditions*

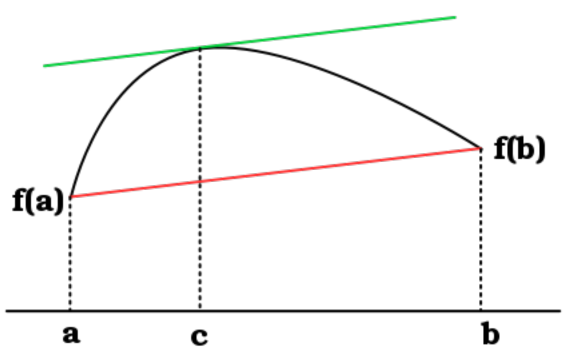
1. $f(x)$ *is continuous on* $[a, b]$.
2. $f(x)$ *is differentiable on* $(a, b)$.

*Then there exists* $c \in (a, b)$*, such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

■

**Remark**: If $f(a) = f(b)$, then $\frac{f(b)-f(a)}{b-a} = 0$. So Rolle's theorem is a special case for mean value theorem.




*Proof.* Let

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

The $g(x)$ satisfies the conditions of Rolle's theorem:

1. $g(x)$ is continuous on $[a, b]$.
2. $g(x)$ is differentiable on $(a, b)$.

3.
$$g(a) = f(a) - f(a) - \frac{f(b) - f(a)}{b - a}(a - a) = 0$$

and

$$g(b) = f(b) - f(a) - \frac{f(b) - f(a)}{b - a}(b - a) = f(b) - f(a) - (f(b) - f(a)) = 0.$$

Hence $g(a) = g(b)$.

Therefore by Rolle's theorem, there exists $c \in (a, b)$ such that $g(c) = 0$, i.e.

$$f'(c) - \frac{f(b) - f(a)}{b - a} = 0$$
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

□

**Example 1.2.** $a = 1$, $b = 2$, $f(x) = x^2$. *Then there exists* $c \in (1, 2)$ *such that*

$$f'(c) = \frac{2^2 - 1^2}{2 - 1} = 3.$$

*In fact,* $2c = 3$, *so* $c = 1.5$. ■

**Theorem 1.4** (Cauchy's mean value theorem)**.** *Suppose* $f, g$ *are functions on* $[a, b]$ *and satisfies the following conditions*

1. $f(x)$, $g(x)$ *are continuous on* $[a, b]$.
2. $f(x)$, $g(x)$ *are differentiable on* $(a, b)$.
3. $g(a) \neq g(b)$.
4. $g'(c) \neq 0$ *for* $c \in (a, b)$.

*Then there exists* $c \in (a, b)$, *such that*

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

■

*Proof.* Let

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a)).$$

Then $h$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Obviously $h(a) = h(b) = 0$. Hence there exists $c \in (a, b)$ such that

$$h'(c) = 0$$
$$f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)}g'(c) = 0$$
$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

□

## 2 Applications

**Example 2.1.** *Show that*

$$|\sin x - \sin y| \le |x - y|$$

■

**Answer.** When $x = y$, the result is trivial. Suppose $x > y$. By the mean value theorem, there exists $c \in (y, x)$ such that

$$\frac{\sin x - \sin y}{x - y} = f'(c) = \cos c$$

$$\sin x - \sin y = \cos c(x - y).$$

Therefore

$$|\sin x - \sin y| = |\cos c||x - y| \le |x - y|.$$

■

**Example 2.2.** *Let $n > 0$. Prove the following inequality*

$$\frac{1}{2\sqrt{n+1}} \le \sqrt{n+1} - \sqrt{n} \le \frac{1}{2\sqrt{n}}.$$

■

**Answer.** In the mean value theorem, let $f(x) = \sqrt{x}$, $a = n$, $b = n+1$, then there exists $c \in (n, n+1)$ such that

$$\sqrt{n+1} - \sqrt{n} = f'(c) = \frac{1}{2\sqrt{c}}.$$

Because $c \in (n, n+1)$,

$$\frac{1}{2\sqrt{n+1}} \le \frac{1}{2\sqrt{c}} \le \frac{1}{2\sqrt{n}}.$$

So

$$\frac{1}{2\sqrt{n+1}} \le \sqrt{n+1} - \sqrt{n} \le \frac{1}{2\sqrt{n}}.$$

■

**Proposition 2.1.** *If $f(x)$ is differentiable on $(a, b)$ and $f'(x) = 0$ for all $x \in (a, b)$. Then $f$ is a constant function.* ■

*Proof.* Let $x_0 \in (a, b)$. For any $x \in (a, b)$, by the the mean value theorem there exists $c$ between $x_0$ and $x$ such that

$$\frac{f(x) - f(x_0)}{x - x_0} = f'(c) = 0.$$

Hence $f(x) = f(x_0)$ for any $x \in (a, b)$. So $f(x)$ is a constant function. □

**Proposition 2.2.** *Suppose $f(x)$ is continuous on $[a, b]$, differentiable on $(a, b)$. Suppose further that $f'(x) > 0$ (resp. $f'(x) < 0$) on $(a, b)$, then $f(x)$ is a strictly increasing (resp. decreasing) function.* ■

*Proof.* Suppose $f'(x)$ for $x \in (a, b)$. Let $x_1, x_2 \in [a, b]$ and suppose $x_1 < x_2$. Then by the mean value theorem, there exists $c \in (x_1, x_2)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) > 0.$$

Hence

$$f(x_2) - f(x_1) > 0$$
$$f(x_2) > f(x_1).$$

□