2017-18 MATH1010
Lecture 13: Implicit functions and differentiation.
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1 Implicit function

The function we have worked with so far have all been given by equations of the form $y = f(x)$ in which the dependent variable y on the left is given explicitly by an expression on the right involving the independent variable x. A function in this form is said to be in explicit form. For example, the functions

$$y = x^2 + 4x \qquad y = \frac{x^4 + 3}{x^2 - 1} \qquad \text{or} \qquad y = \sqrt{1 - x^2}$$

are all functions in explicit form.

Unfortunately, certain equations in x and y, such as

$$x^2y^3 + 5y^2 = 1 - x^3 \qquad \text{or} \qquad x^2y - 6 = 3x + 2y$$

either cannot be solved explicitly for y in terms of x or can be done so only with great effort.

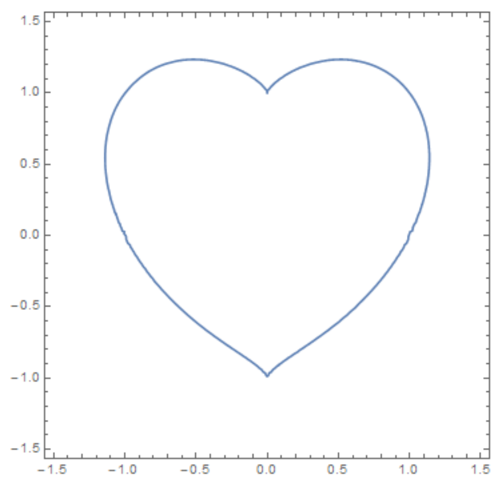


Figure 1: Mathematician's love: $(x^2 + y^2 - 1)^3 = x^2y^3$. `wolframalpha.com: plot (x^2 + y^2 - 1)^3 = x^2 y^3`

Example 1.1. *Discuss the curve $x^2 + y^2 = 25$.* ■

Answer. Solve y in terms of x, we have $y = \pm\sqrt{25 - x^2}$ for $x \in [-5, 5]$. We see from the formula that y is not uniquely determined by x. So we cannot say y is a function of x. Refer to the graph of example 2.2.

1. Consider $(3, 4)$ on the curve and restrict to a small neighbourhood of the point $(3, 4)$ on the curve. y can be uniquely given by $y = \sqrt{25 - x^2}$.
2. Similarly if we consider the neighbourhood of $(3, -4)$ on the curve, then y can be given by $y = -\sqrt{25 - x^2}$.

3. However, if we consider the neighbourhood of $(5, 0)$ on the curve, then y cannot be expressed by a function of x.

■

Observation.

1. For some points (x_0, y_0) on the curve, we can find a small neighbourhood, such that the curve inside the neighbourhood is actually a graph. So y can be given by $y = f(x)$ when x is in a small neighbourhood of x_0.
2. But for some point (x_0, y_0) on the curve, we **cannot** find a small neighbourhood of (x_0, y_0) such that the curve inside the neighbourhood is a graph.

Theorem 1.1. *(***can be skipped, read the short version below***) Suppose $F(x, y)$ is a function with two variables. Let C be the curve $F(x, y) = 0$. Suppose (x_0, y_0) is in C, i.e., $F(x_0, y_0) = 0$, Except for few exceptions, there exists a small neighbourhood of (x_0, y_0), such that the intersection of the curve and the neighbourhood is a graph. That is, if x is in a small neighborhood of x_0, then $y = f(x)$ is on the curve C:*

$$F(x, f(x)) = 0.$$

■

Short version of the above theorem
Let $F(x, y)$ be a functions (usually polynomials). Let C be the curve $F(x, y) = 0$. We can assume y can be expressed as $f(x)$, i.e. $F(x, f(x)) = 0$. ■

2 Implicit differentiation

Suppose you have an equation that defines y implicitly in terms of x and you want to find the derivative $\frac{dy}{dx}$, how should you proceed? The answer is provided by a method called **implicit differentiation**, which consists of differentiating both sides of the defining equation with respect to x and then solving algebraically for $\frac{dy}{dx}$.

Example 2.1. *Suppose $y = f(x)$ is a differentiable function of x that satisfies the equation $x^2y + y^2 = x^3$. Find the derivative $\frac{dy}{dx}$.* ■

Answer. You are going to differentiate both sides of the given equation with respect to x. So that you will not forget that y is actually a function of x. Temporarily replace y by $f(x)$and begin by rewriting the equation as

$$x^2 f(x) + (f(x))^2 = x^3.$$

Now differentiate both sides of this equation term by term with respect to x:

$$\begin{aligned} \frac{d}{dx}[x^2 f(x) + (f(x))^2] &= \frac{d}{dx}[x^3] \\ \left[x^2\frac{df}{dx} + f(x)\frac{d}{dx}(x^2)\right] + 2f(x)\frac{df}{dx} &= 3x^2 \end{aligned} \tag{1}$$

Thus, we have

$$\begin{aligned}
x^2\frac{df}{dx} + f(x)(2x) + 2f(x)\frac{df}{dx} &= 3x^2 \\
[x^2 + 2f(x)]\frac{df}{dx} &= 3x^2 - 2xf(x) \\
\frac{dy}{dx} &= \frac{3x^2 - 2xf(x)}{x^2 + 2f(x)}.
\end{aligned} \tag{2}$$

Finally, replace $f(x)$ by y to get

$$\frac{dy}{dx} = \frac{3x^2 - 2xy}{x^2 + 2y}.$$

■

Remark: Temporarily replacing y by $f(x)$ as in Example 2.1 is a useful device for illustrating the implicit differentiation process, but as soon as you feel comfortable with the technique, try to leave out this unnecessary step and differentiate across the equation directly. Just keep in mind that y is really a function of x, and remember to use the chain rule when it is appropriate.

Implicit Differentiation Procedure

Suppose an equation defines y implicitly as a differentiable function of x. To find $\dfrac{dy}{dx}$:

1. Differentiate both sides of the equation with respect to x. Remember that y is really a function of x, and use the chain rule when differentiating terms containing y.

2. Solve the differentiated equation algebraically for $\dfrac{dy}{dx}$ in terms of x and y.

Example 2.2. *Find the slope of the tangent line to the circle $x^2 + y^2 = 25$ at the point $(3, 4)$. What is the slope at the point $(3, -4)$?* ■

Answer. Differentiating both sides of the equation $x^2 + y^2 = 25$ with respect to x, you get

$$\begin{aligned}
2x + 2y\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= -\frac{x}{y}
\end{aligned}$$

The slope at $(3, 4)$ is the value of the derivative when $x = 3$ and $y = 4$:

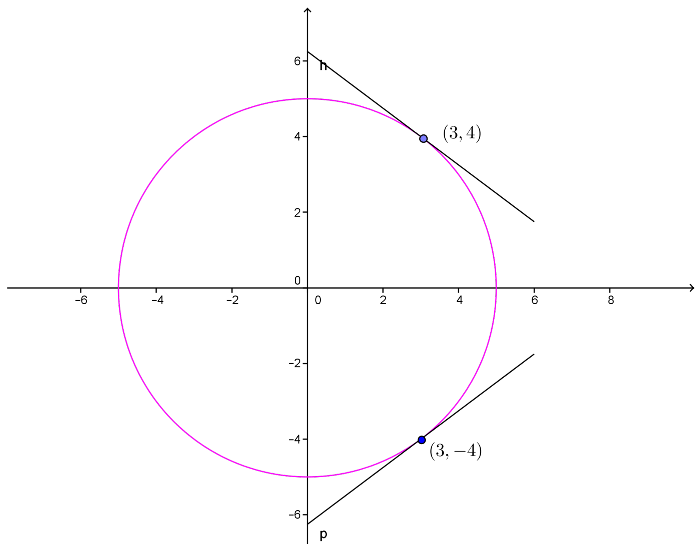
$$\left.\frac{dy}{dx}\right|_{(3,4)} = -\left.\frac{x}{y}\right|_{\substack{x=3\\y=4}} = -\frac{3}{4}$$

Similarly, the slope at $(3, -4)$ is the value of $\dfrac{dy}{dx}$ when $x = 3$ and $y = -4$:

$$\left.\frac{dy}{dx}\right|_{(3,-4)} = -\left.\frac{x}{y}\right|_{\substack{x=3\\y=-4}} = -\frac{3}{-4} = \frac{3}{4}$$

The graph of the circle is shown together with the tangent lines at $(3, 4)$ and $(3, -4)$.

■



Example 2.3. *Consider the curve defined by*

$$x^3 + y^3 = 9xy.$$

1. *Compute* $\frac{dy}{dx}$.
2. *Find the slope of the tangent line at* $(4, 2)$.

■

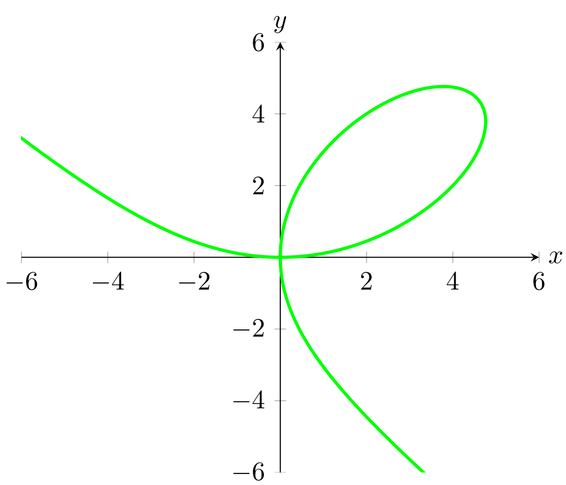


Figure 2: A plot of $x^3 + y^3 = 9xy$. While this is not a function of y in terms of x, the equation still defines a relation between x and y.

Answer. Starting with

$$x^3 + y^3 = 9xy,$$

we apply the differential operator $\frac{d}{dx}$ to both sides of the equation to obtain

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx} 9xy.$$

Applying the sum rule we see

$$\frac{d}{dx}x^3 + \frac{d}{dx}y^3 = \frac{d}{dx} 9xy.$$

Let's examine each of these terms in turn. To start

$$\frac{d}{dx}x^3 = 3x^2.$$

On the other hand $\frac{d}{dx}y^3$ is somewhat different. Here you imagine that $y = y(x)$, and hence by the chain rule

$$\begin{aligned}\frac{d}{dx}y^3 &= \frac{d}{dx}(y(x))^3 \\ &= 3(y(x))^2 \cdot y'(x) \\ &= 3y^2\frac{dy}{dx}.\end{aligned}$$

Considering the final term $\frac{d}{dx}9xy$, we again imagine that $y = y(x)$. Hence

$$\begin{aligned}\frac{d}{dx}9xy &= 9\frac{d}{dx}x \cdot y(x) \\ &= 9\left(x \cdot y'(x) + y(x)\right) \\ &= 9x\frac{dy}{dx} + 9y.\end{aligned}$$

Putting this all together we are left with the equation

$$3x^2 + 3y^2\frac{dy}{dx} = 9x\frac{dy}{dx} + 9y.$$

At this point, we solve for $\frac{dy}{dx}$. Write

$$\begin{aligned}3x^2 + 3y^2\frac{dy}{dx} &= 9x\frac{dy}{dx} + 9y \\ 3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} &= 9y - 3x^2 \\ \frac{dy}{dx}\left(3y^2 - 9x\right) &= 9y - 3x^2 \\ \frac{dy}{dx} &= \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3y - x^2}{y^2 - 3x}.\end{aligned}$$

For the second part of the problem, we simply plug $x = 4$ and $y = 2$ into the formula above, hence the slope of the tangent line at $(4, 2)$ is $\frac{5}{4}$, see Figure 3. ■

You might think that the step in which we solve for $\frac{dy}{dx}$ could sometimes be difficult—after all, we're using implicit differentiation here instead of the more difficult task of solving the equation $x^3 + y^3 = 9xy$ for y, so maybe there are functions where after taking the derivative we obtain something where it is hard to solve for $\frac{dy}{dx}$. In fact, *this never happens*. All occurrences $\frac{dy}{dx}$ arise from applying the chain rule, and whenever the chain rule is used it deposits a single $\frac{dy}{dx}$ multiplied by some other expression. Hence our expression is linear in $\frac{dy}{dx}$, it will always be possible to group the terms containing $\frac{dy}{dx}$ together and factor out the $\frac{dy}{dx}$, just as in the previous example.

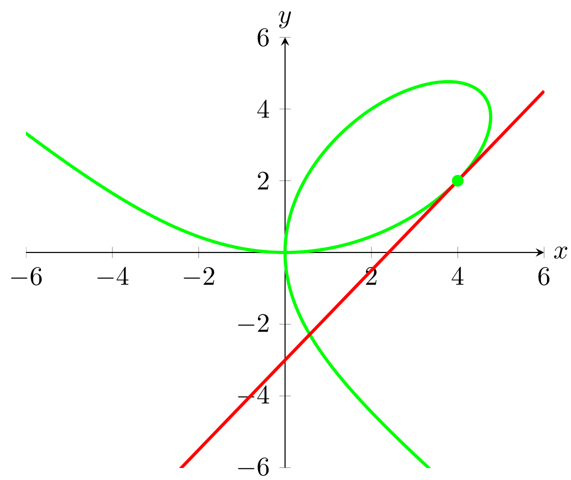


Figure 3: A plot of $x^3 + y^3 = 9xy$ along with the tangent line at $(4, 2)$.