2017-18 MATH1010
Lecture 11: Higher derivatives
Charles Li

1 Higher derivatives

Higher derivatives are obtained by repeatedly differentiating a function $f(x)$. Example, if $f'(x)$ is differentiable, then the derivative of $f'(x)$, denoted by $f'(x)$ is given by

$$f''(x) = \frac{d}{dx} f'(x).$$

or

$$\frac{d^2 f}{dx^2} = \frac{d}{dx} f'(x).$$

The process can be continued, provided the derivative exists. So for example, the derivative of $f''(x)$ can be denoted by $f'''(x)$, or $f^{(3)}(x)$ or $\frac{d^3 f}{dx^3}$. More generally, the nth derivative $f(x)$ is the derivative of the $(n-1)$st derivative. We call $f(x)$ the zeroth derivative. In summary

$$f^{(n)}(x) = \frac{d}{dx} f^{(n-1)}(x)$$

can also be denoted by

$$\frac{d^n}{dx^n} f(x).$$

If $y = f(x)$, then the nth derivative of y can be denoted by

$$y^{(n)} \text{ or } \frac{d^n y}{dx^n}.$$

The first, second and the third derivative can be written as y', y'' and y''' respectively.

Example 1.1. *$f(x) = 2x^3 - 3x^2 + x - 1$, find $f'(x)$, $f''(x)$, $f'''(x)$, $f^{(4)}(x)$, ...* ■

Answer.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left(2x^3 - 3x^2 + x - 1\right) = 6x^2 - 6x + 1, \\
f''(x) &= \frac{d}{dx}\left(6x^2 - 6x + 1\right) = 12x - 6, \\
f'''(x) &= \frac{d}{dx}(12x - 6) = 12, \\
f^{(4)}(x) &= \frac{d}{dx} 12 = 0.
\end{aligned}$$

Generally $f^{(n)}(x) = 0$ for $n \geq 4$.

Example 1.2. *Let $f(x)$ be a polynomial for degree k. Generally $f^{(n)}(x) = 0$ for $n > k$.* ■

Example 1.3. *Let*

$$f(x) = x^5 + 2x - \frac{3}{x}.$$

Compute $f'''(x)$. ■

Answer.

$$f'(x) = \frac{d}{dx}\left(x^5 + 2x - \frac{3}{x}\right) = 5x^4 + 2 + \frac{3}{x^2},$$
$$f''(x)\frac{d}{dx}\left(5x^4 + 2 + \frac{3}{x^2}\right) = 20x^3 - \frac{6}{x^3},$$
$$f'''(x)\frac{d}{dx}\left(20x^3 - \frac{6}{x^3}\right) = 60x^2 + \frac{18}{x^4}.$$

Example 1.4. *Let $f(x) = \frac{1}{x}$, find $f^{(n)}(x)$.* ■

Answer. $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$, $f'''(x) = -2 \times 3 \times x^{-4}$, $f^{(4)}(x) = 2 \times 3 \times 4 \times x^{-5}$. Generally

$$f^{(n)}(x) = (-1)^n n! x^{-(n+1)}.$$

Example 1.5. *Suppose $f(x)$ and $g(x)$ are differentiable, find*

$$\frac{d^3}{dx^3} f(g(x)).$$

■

Answer.

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$
$$\frac{d^2}{dx^2} f(g(x)) = \frac{d}{dx} f'(g(x))g'(x)$$
$$= f''(g(x))g'(x)g'(x) + f'(g(x))g''(x)$$
$$= f''(g(x))(g'(x))^2 + f'(g(x))g''(x).$$
$$\frac{d^3}{dx^3} f(g(x)) = \frac{d}{dx}(f''(g(x))(g'(x))^2 + f'(g(x))g''(x))$$
$$= f'''(g(x))g'(x)(g'(x))^2 + f''(g(x))(2g'(x))g''(x) + f''(g(x))g'(x)g''(x) + f'(g(x))g'''(x).$$
$$= f'''(g(x))(g'(x))^3 + 3f''(g(x))g'(x)g''(x) + f'(g(x))g'''(x).$$

Example 1.6. *Let $f(x) = e^x$, find $f^{(n)}(x)$.* ■

Answer.

$$f'(x) = \frac{d}{dx} e^x = e^x,$$
$$f''(x) = \frac{d}{dx} e^x = e^x,$$
$$f'''(x) = \frac{d}{dx} e^x = e^x.$$

Generally $f^{(n)}(x) = e^x$.

Example 1.7. *Let $f(x) = xe^x$, find $f^{(n)}(x)$.* ■

Answer.

$$\begin{aligned} f'(x) &= \frac{d}{dx} x e^x = e^x + x e^x = (x+1)e^x, \\ f''(x) &= \frac{d}{dx}(x+1)e^x = e^x + (x+1)e^x = (x+2)e^x, \\ f'''(x) &= \frac{d}{dx}(x+2)e^x = e^x + (x+2)e^x = (x+3)e^x. \end{aligned}$$

Generally

$$f^{(n)}(x) = (x+n)e^x.$$

Proposition 1.1. *Let $f(x)$ and $g(x)$ be function. Suppose the nth derivative of $f(x)$ and $g(x)$ exists. Then*

$$\begin{aligned} \frac{d^n}{dx^n} f(x)g(x) = {}& {}_nC_0 f^{(n)}(x)g(x) + {}_nC_1 f^{(n-1)}(x)g^{(1)}(x) + {}_nC_2 f^{(n-2)}(x)g^{(2)}(x) \\ & + \cdots + {}_nC_{n-2} f^{(2)}(x)g^{(n-2)}(x) + {}_nC_{n-1} f^{(1)}(x)g^{(n-1)} + {}_nC_n f(x)g^{(n)}(x), \end{aligned}$$

where

$${}_nC_r = \frac{n!}{r!(n-r)!},$$

which are the coefficients of the binomial expansion. ■

Proof. By induction on n. Skipped. □

Example 1.8. *Compute*

$$\frac{d^n}{dx} x^2 e^x.$$

■

Answer. Because $\frac{d^k}{dx^k} x^2 = 0$ for $k \geq 3$,

$$\begin{aligned} \frac{d^n}{dx} x^2 e^x &= x^2 \frac{d^n}{dx^n} e^x + {}_nC_1 \left(\frac{d}{dx} x^2\right)\left(\frac{d^{n-1}}{dx^{n-1}} e^x\right) + {}_nC_2 \left(\frac{d^2}{d^2 x} x^2\right)\left(\frac{d^{n-2}}{dx^{n-2}} e^x\right) \\ &= x^2 e^x + 2nxe^x + n(n-1)e^x \\ &= (x^2 + 2nx + n(n-1))e^x. \end{aligned}$$