2018 MATH1010
Lecture 1: Notation and functions
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The lecture note was used during 2016-17 Term 1. It is for reference only. It may contain typos. Read at your own risk.

1 Notation: set

- **Set** is a collection of **objects** (also called **elements**)
- $\subseteq$: subset of. $A \subseteq B$ means A is a subset of B.
- $\in$: belongs to. $a \in A$ means a belongs to A. We also say that a is an objects (or elements) of A.
- $\cup$: Union, $A \cup B$ means the union of the sets A and B.
- $\cap$: intersection, $A \cap B$ means the intersection of the sets A and B.
- $A \backslash B$: elements in A but not in B.
- $\emptyset$ represent an empty set.
- $\mathbf{Z}$ represent the set of integers, $\mathbf{R}$ denotes the set of real numbers.

Example 1

1. $A = \{1, 2, 3\}$, $B = \{2, 3, 5\}$ $C = \{1, 2, 3, 4, 5\}$. Then $A \subset C$, $1 \in A$, $A \cup B = \{1, 2, 3, 5\}$, $A \cap B = \{2, 3\}$, $A \backslash B = \{1\}$, $1 \notin B$ and $B \not\subseteq A$
2. C = all students studying at CUHK. M = all math major students currently studying at CUHK. Then $M \subseteq C$. You $\in C$.

2 Notations: intervals

- $\mathbf{R}$: the set of all real numbers.

- $[a, b]$: the set of all real numbers x such that $a \leq x \leq b$.
- (a, b): the set of all real numbers x such that $a < x < b$.
- $(a, b]$: the set of all real numbers x such that $a < x \leq b$.
- $[a, b)$: the set of all real numbers x such that $a \leq x < b$.
- $[a, \infty)$: the set of all real numbers x such that $a \leq x$.
- $\mathbf{R} \backslash \{a\}$: the set of all real numbers x, except $x = a$.

Example 2

1. Solve $x^2 < 1$.

2. Solve $x^2 > 1$.

Answer:

1. $x^2 < 1 \Leftrightarrow -1 < x < 1$. So the answer is $(-1, 1)$.
2. If $x^2 > 1$, then $x < -1$ or $x > 1$. So the answer is $(-\infty, -1) \cup (1, \infty)$.

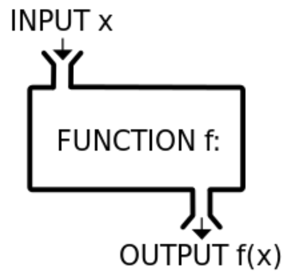
Exercises

1. Write down the meanings of the following sets.
   (a) $(-\infty, a)$.
   (b) $\mathbf{R} \backslash \{1, 2, 3\}$
   (c) $\mathbf{R} \backslash [2, 3)$.
2. Show that $\mathbf{R} \backslash [1, \infty) = (-\infty, 1)$.

3 Functions

Definition 1 *A* **function** *is a rule that assigns to each object in a set A exactly one object in a set B.*

The set A is called the **domain** *of the function.*
The set B of assigned objects in B is called the **range**.



Example 3 $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2 + 4$.
Then
$$f(-3) = (-3)^2 + 4 = 13$$
.
Or
$$y = x^2 + 4.$$
Then x is called an **independent variable** and y (which depends on x) is called a **dependent variable**

Example 4 Find the domain of the functions.
a. $f(x) = \frac{1}{x-3}$.
b. $g(t) = \frac{\sqrt{3-2t}}{t^2+4}$.
Answer: a. x can be any real number except when the denominator is 0, i.e. $x = 3$. So the domain is $\mathbf{R}\backslash\{3\}$.
b. Only non-negative numbers have square roots. So the domain consists of t such that $3 - 2t \geq 0$,i.e., $t \leq \frac{3}{2}$ and hence the domain is $(-\infty, \frac{3}{2}]$.

Example 5 What is the difference between $f(x) = \frac{x^2-1}{x-1}$ and $g(x) = x + 1$.
By the difference of two squares: $x^2 - 1 = (x - 1)(x + 1)$. So $f(x) = \frac{(x-1)(x+1)}{x-1} = x + 1$. The problem is that we can't cancel out the factor $x - 1$ when $x - 1 = 0$.
If fact, the main difference is the domain:
The domain of $f(x)$ is $\mathbf{R}\backslash\{1\}$.
The domain of $g(x)$ is $\mathbf{R}$.

Definition 2 *A piecewise function is defined by using more than one formula, with each individual formula defined describe the func-*

tion on a subset of the domain.

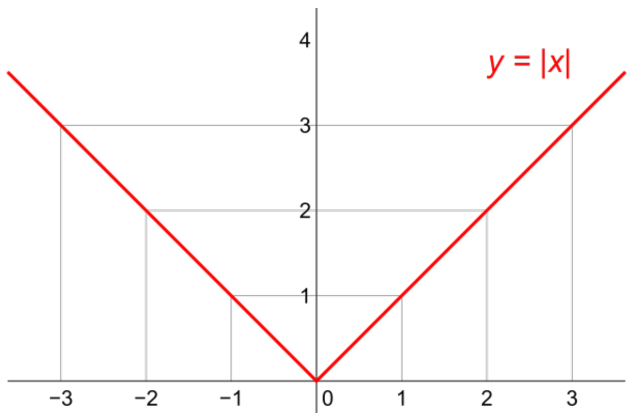
Example 6 $f : \mathbf{R} \to \mathbf{R}$ *is defined by*

$$f(x) = \begin{cases} 1 & \textit{if } x < 0, \\ 2x & \textit{if } x \geq 0. \end{cases}$$

Then $f(-1) = 1,\ f(0) = 0$ *and* $f(1) = 2$.

Example 7 Absolute value

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$



Example 8 Let $f : \mathbf{R}\backslash\{1\}$ is defined by $f(x) = \frac{x^2-1}{x-1}$. We can write the function as

$$f(x) = \begin{cases} x + 1 & \text{if } x \neq 1, \\ \text{undefined} & \text{if } x = 1. \end{cases}$$

Example 9 Let $f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q},\ p, q \text{ relatively prime,} \\ 0 & \text{otherwise.} \end{cases}$$

Example 10

$$f(x) = \begin{cases} 1 & \text{if } 1 \leq x \leq 2, \\ 0 & \text{if } x > 10, \\ -1 & \text{otherwise.} \end{cases}$$

4 Graphs of functions

If f is a function with domain D, then the **graph** of the function f are points in the Cartesian plan $(x, f(x))$. In set theory notation, the graph is

$$\{(x, f(x)) \,|\, x \in D\}$$

Example 11 If $f(x)$ is a linear equation, i.e, $f(x) = ax + b$, then the graph is a straight line.

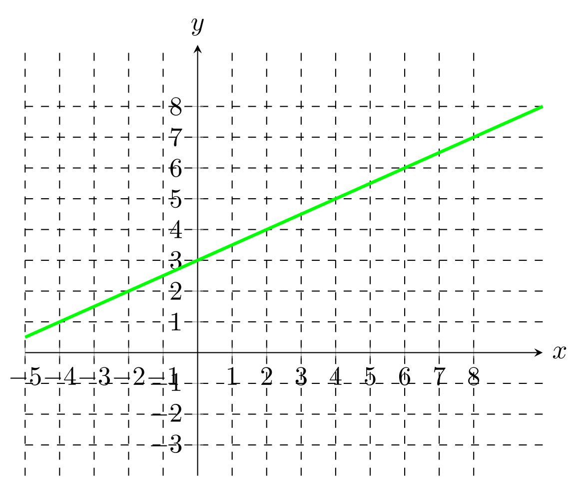


Figure 1: A plot of $y = \frac{x}{2} + 3$.

Example 12 $f(x) = \frac{1}{x}$

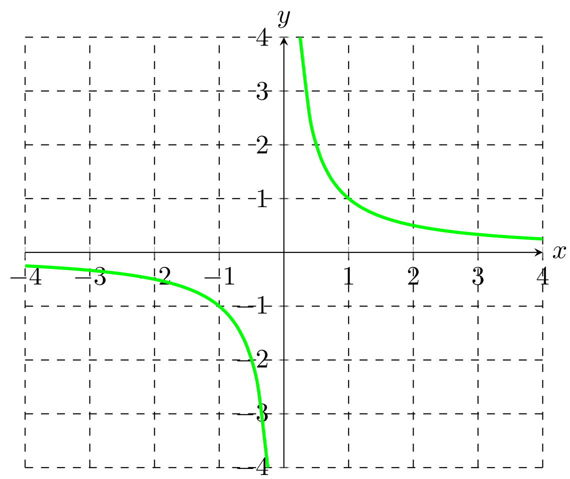


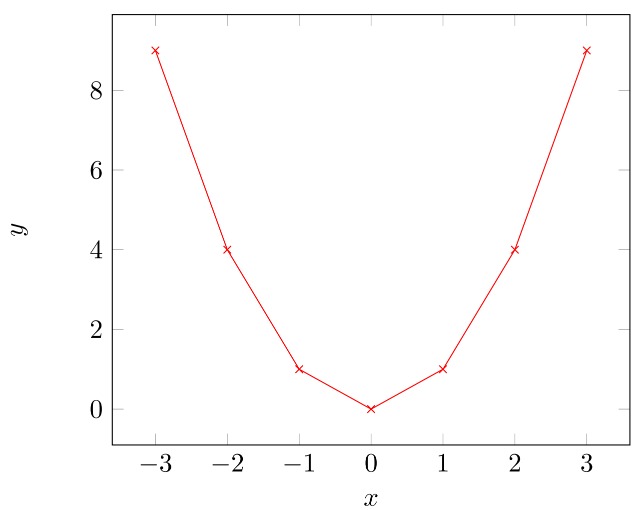
Figure 2: A plot of $y = \frac{1}{x}$.

4.1 Basic technique of plotting a graph

Example 13 Graph $y = x^2$.
Make a table of xy-pair:

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

Then join all the points together.



4.2 The vertical line test for a function

Not every curve can be a graph of a function. A function can have only one value $f(x)$ for each x in its domain, so no vertical line can intersect the graph of a function more than once. If a is in the domain of the function f, then the vertical line $x = a$ will intersect the graph of f at the single point $(a, f(a))$.

Example 14 Plot all the points $x^2 - 2x + y^2 = 0$. Explain why it is not a graph of any function.
Answer $x^2 - 2x + y^2 = 0$ if and only if $(x - 1)^2 + y^2 = 1$. So it is a circle, with center at $(1, 0)$ and radius$= 1$.
Let $x = a = 1$ (or for any $x \neq 0, 2$). Then $y = \pm 1$. So $x = 1$ corresponding to two y's. Hence it is not a graph.
How about $x^2 - 2x + y^2 = 0$ and $y \geq 0$?