Tutorial 10

5.5.3 (a) Suppose that f is a function of moderate decrease on $\mathbb{R}$ whose Fourier transform $\hat{f}$ is continuous and satisfies

$$\hat{f}(\xi) = O\left(\frac{1}{|\xi|^{1+\alpha}}\right) \quad \text{as } |\xi| \to \infty$$

for some $0<\alpha<1$. Prove that f is Hölder continuous of order α.

(b) Let f be a continuous function on $\mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x=0 \text{ or } |x|\geq 1 \\ 1/\log(1/|x|) & \text{if } 0<|x|\leq 1/2 \\ \text{linear} & \text{otherwise.} \end{cases}$$

Show that $\hat{f}$ is not of moderate decrease.

<u>Rm:</u> Fourier transform of a moderately decreasing func. may not be of moderate decrease.

<u>Proof</u> (a) For any $x\in\mathbb{R}$, $h\in\mathbb{R}$, by the Fourier inversion formula

$$|f(x+h)-f(x)| = \left|\int_{-\infty}^{\infty} \hat{f}(\xi)\left(e^{2\pi i(x+h)\xi} - e^{2\pi i x\xi}\right)d\xi\right|$$

$$\overset{\hat{f}(\xi)=O\left(\frac{1}{|\xi|^{1+\alpha}}\right)}{\leq} M\int_{-\infty}^{\infty} \frac{1}{|\xi|^{1+\alpha}}|e^{2\pi i x\xi}||e^{2\pi i h\xi}-1|\,d\xi \quad \text{for some } M>0$$

$$= M\left(\int_{|\xi|\leq 1/|h|} \frac{|e^{2\pi i h\xi}-1|}{|\xi|^{1+\alpha}} + \int_{|\xi|\geq \frac{1}{|h|}} \frac{2}{|\xi|^{1+\alpha}}\,d\xi\right)$$

$$\overset{\text{Mean Value Theorem}}{\leq} M\left(\int_{|\xi|\leq 1/|h|} |h|\,|\xi|^{-\alpha} + \left(-\frac{4}{\alpha}|\xi|^{-\alpha}\right)\Big|_{1/|h|}^{\infty}\right)$$

$$\leq M\left(\frac{2|h|}{1-\alpha}|\xi|^{1-\alpha}\Big|_0^{1/|h|}+\frac{4}{\alpha}|h|^{\alpha}\right)$$

$$=\frac{2M(2-\alpha)}{(1-\alpha)\alpha}|h|^{\alpha}.$$

Therefore, f is Hölder cont. of order α.

(b) Since $\lim_{x\to 0}\left|\frac{1}{\log(1/|x|)}-0\Big/|x|^{\alpha}\right|=\lim_{y\to+\infty}\frac{y^{\alpha}}{\log y}$

$$\overset{\text{L'Hospital}}{=}\lim_{y\to+\infty}\frac{\alpha y^{\alpha-1}}{\frac{1}{y}}=\infty,\quad \text{for any } 0<\alpha<1$$

it follows from the result of (a) that

there is no $\varepsilon>0$ such that $\hat{f}(\xi)=O\left(\frac{1}{|\xi|^{1+\varepsilon}}\right)$ as $|\xi|\to\infty$.

Therefore, $\hat{f}$ is not of moderate decrease. □

<u>Ex2.</u> Find the Fourier transform of the following functions:

(a) $f(x)=e^{-x}H(x)$, where $H(x):=\chi_{[0,\infty)}$ is the Heaviside func.

(b) $g(x)=e^{-|x|}$

(c) $h(x)=e^{-a|x|}$, $a>0$

(d) $k(x)=e^{-|x|}\cos x$

<u>Sol.:</u> (a) $\hat{f}(\xi)=\int_0^{\infty}e^{-x}e^{-2\pi i x\xi}\,dx=-\frac{1}{2\pi i\xi+1}e^{-(2\pi i\xi+1)x}\Big|_0^{\infty}$

$$=\frac{1}{2\pi i\xi+1}$$

(b) $\hat{g}(\xi)=\int_0^{\infty}e^{-x}e^{-2\pi i x\xi}\,dx+\int_{-\infty}^{0}e^{x}e^{-2\pi i x\xi}\,dx$

$$= \hat{f}(\xi) + \frac{1}{-2\pi i\xi+1} e^{(-2\pi i\xi+1)x} \Big|_{-\infty}^{0}$$

$$= \frac{1}{2\pi i\xi+1} + \frac{1}{-2\pi i\xi+1} = \frac{2}{(2\pi\xi)^2+1} = \frac{2}{1+4\pi^2\xi^2}$$

(c) $\hat{h}(\xi) = a^{-1}\hat{g}(a^{-1}\xi) = \frac{2}{a(1+4\pi^2 a^{-2}\xi^2)} = \frac{2a}{a^2+4\pi^2\xi^2}$

(d) $$\hat{k}(\xi) = \int_0^\infty \frac{e^{-x+ix}+e^{-x-ix}}{2} e^{-2\pi i x\xi}\,dx + \int_{-\infty}^0 \frac{e^{x+ix}+e^{x-ix}}{2} e^{-2\pi i x\xi}\,dx$$

$$= \frac{1}{1+(2\pi\xi+1)i} + \frac{1}{1+(2\pi\xi-1)i} + \frac{1}{1-(2\pi\xi-1)i} + \frac{1}{1-(2\pi\xi+1)i}$$

$$= \frac{2}{1+(2\pi\xi+1)^2} + \frac{2}{1+(2\pi\xi-1)^2} = \frac{2(2\pi^2\xi^2+1)}{4\pi^4\xi^4+1}$$ □

Ex3. Find f s.t. it satisfies the integral equation

$$\int_{-\infty}^{\infty} f(x-y)\,e^{-|y|}\,dy = 2e^{-|x|} - e^{-2|x|}.$$

(Apply Fourier transform and the properties of convolution.)

Sol. Define $h(x) = \int_{-\infty}^{\infty} f(x-y)\,e^{-|y|}\,dy = 2e^{-|x|} - e^{-2|x|}$.

Direct calculation gives

$$\hat{h}(\xi) = \frac{4}{1+4\pi^2\xi^2} - \frac{2\cdot 2}{2^2+4\pi^2\xi^2} = \frac{12}{(1+4\pi^2\xi^2)(4+4\pi^2\xi^2)}$$

On the other hand,

$$\hat{h}(\xi) = \hat{f}\hat{g} = \hat{f}\,\frac{2}{1+4\pi^2\xi^2}.$$

Therefore. $\hat{f} = \frac{6}{4+4\pi^2\xi^2}$ and $f = \frac{3}{2}e^{-2|x|}$ □