Tutorial 9

On the mid-term

1. $$f(x) \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$
$$= a_0 + \sum_{n=1}^{\infty} \left(a_n \frac{e^{inx}+e^{-inx}}{2} + b_n \frac{e^{inx}-e^{-inx}}{2i} \right)$$
$$\sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$\Rightarrow \quad a_0 = c_0, \quad a_n = c_n + c_{-n}, \quad b_n = i(c_n - c_{-n}) \quad n \geq 1$$

or
$$c_n = \begin{cases} \frac{a_n}{2} + \frac{b_n}{2i} & n \geq 1; \\ a_0 & n = 0; \\ \frac{a_{-n}}{2} - \frac{b_{-n}}{2i} & n \leq -1. \end{cases}$$

(Different books may have different coefficients in the definitions. So be careful and be consistent with the def. in the problem.)

2. $f(x) = e^x$ on $[-\pi, \pi]$. (a) Determine the Fourier series on f on $[-\pi, \pi]$; (b) Calculate $\sum_{n=-\infty}^{\infty} \frac{1}{n^2+1}$.

Sol. (a) $f \sim \cancel{\frac{\pi}{2} + \sum_{n \neq 0} \frac{(-1)^n - 1}{\pi n^2} e^{inx}}$
$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{2\pi(1-in)} (e^{\pi} - e^{-\pi}) e^{in\pi}$$

(b) $$f(\cancel{x}) = (e^{\pi} - e^{-\pi}) \left[\frac{1}{2\pi} + \frac{1}{2\pi} \sum_{n=1}^{\infty} \left(\frac{1}{1-in} + \frac{1}{1+in} \right) \right]$$
$$= \frac{e^{\pi} - e^{-\pi}}{2\pi} \sum_{n=-\infty}^{\infty} \frac{1}{n^2+1}$$

Since f is not continuous at $x = \pi$.

It follows from the Parseval identity that

$$\left(\frac{e^{\pi}-e^{-\pi}}{2\pi}\right)^2 \sum_{n=-\infty}^{\infty} \frac{1}{|1-in|^2} = \frac{1}{2\pi}\,\frac{e^{2\pi}-e^{-2\pi}}{2}$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} = \frac{\pi(e^{\pi}+e^{-\pi})}{e^{\pi}-e^{-\pi}} \qquad \square$$

4. f is Riemann integrable on the circle, $\exists\, c>0$, $\alpha\in(0,1]$, s.t. $|f(x)-f(0)|\leq c|x|^{\alpha}$ for all $x\in[-\pi,\pi)$. Prove that

$$\lim_{N\to\infty} S_N f(0) = f(0).$$

Proof. Note that
$$\left|\frac{1}{2\pi}\int_{-\delta}^{\delta} \frac{\sin\left(\left(N+\frac{1}{2}\right)\theta\right)}{\sin\frac{\theta}{2}}\left(f(\theta)-f(0)\right)d\theta\right|$$

$$\leq \frac{1}{2\pi}\int_{-\delta}^{\delta} \frac{1}{\frac{2}{\pi}\frac{|\theta|}{2}}\, c|\theta|^{\alpha}\, d\theta = \frac{c}{\alpha}\delta^{\alpha}$$

Therefore, for any $\varepsilon>0$, $\exists\, \delta_0$, s.t. $\frac{c}{\alpha}\delta_0^{\alpha} < \frac{\varepsilon}{2}$.

It suffices to prove that $\exists\, N_0$, s.t. $\forall\, n>N_0$,

$$\left|\frac{1}{2\pi}\int_{\delta\leq|\theta|\leq\pi} \frac{\sin\left(\left(N+\frac{1}{2}\right)\theta\right)}{\sin\frac{\theta}{2}}\left(f(\theta)-f(0)\right)\right| < \frac{\varepsilon}{2}. \qquad (*)$$

Since $\left|\chi_{\delta\leq|\theta|\leq\pi}\frac{f(\theta)-f(0)}{\sin\frac{\theta}{2}}\right| \leq c\pi\chi_{\delta\leq|\theta|\leq\pi}|\theta|^{\alpha-1}$ is Riemann integrable on the circle, (*) is proved by the Riemann-Lebesgue Lemma.

$\square$

5. (a) Show that the trigonometric series

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} e^{inx}$$

is the Fourier series of a Riemann integrable function on the circle.

(b) ——— $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} e^{inx}$ ——— NOT ———

(c) ——— $\sum_{n=1}^{\infty} \frac{1}{n} e^{inx}$ ——— NOT ———

Proof: (a) Since the series $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ converges absolutely,

$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} e^{inx}$ defines a continuous function f on the circle.

Moreover, $\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\,dx = \sum_{m=1}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{m\sqrt{m}} e^{imx} e^{-inx}\,dx$

$$= \begin{cases} \frac{1}{n\sqrt{n}} & n \geq 1; \\ 0 & n \leq 0. \end{cases}$$

So f is the Riemann integrable function ~~of~~ which the Fourier series is $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} e^{inx}$.

(b) Parseval identity.

(c) Check the textbook P84.

$\square$