Tutorial 3

- Uniqueness of Fourier series → 1 theorem + 2 corollaries
- Convolutions & Good kernels → Dirichlet kernel X
- Cesàro & Abel summability → Fejér kernel. Poisson kernel ✓

"mollifier in $\mathbb{R}^n$"

$$\varphi(x) = \begin{cases} e^{-\frac{1}{1-x^2}} / I_n & \text{if } |x|<1 \\ 0 & \text{if } |x|>1. \end{cases}$$

where $I_n = \int_{|x|<1} e^{-\frac{1}{1-x^2}}\, dx$, and

$$\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$$

$\Rightarrow$ (a) for all $\varepsilon>0$. $\int_{\mathbb{R}^n} \varphi_\varepsilon(x)\, dx = 1$

(b) ______

(c) For every $\delta>0$, $\int_{\delta<|x|} |\varphi_\varepsilon(x)|\, dx \to 0$ as $\varepsilon \to 0$.

2.6.13 (a) Show that if the series $\sum_{n=1}^{\infty} c_n$ of complex numbers converges to a finite limit s, then the series is Abel summable to s.

(c) Show that if the series $\sum_{n=1}^{\infty} c_n$ is Cesàro summable to σ, then it is Abel summable to σ.

Proof. (a) It suffices to prove the case $s=0$ since if $s \neq 0$, we can consider the series $\left(a_1 - s + \sum_{n=2}^{\infty} c_n\right)$, and if $\left(a_1 - s + \sum_{n=2}^{\infty} c_n\right)$ is Abel summable to 0, then $\sum_{n=1}^{\infty} c_n$ is Abel summable to s.

Now suppose that $s=0$.

$$\sum_{n=1}^{N} c_n r^n = \sum_{n=1}^{N} (s_n - s_{n-1}) r^n = (1-r)\sum_{n=1}^{N-1} s_n r^n + s_N r^n.$$

Let $N \to \infty$, we have

$$\sum_{n=1}^{\infty} c_n r^n = (1-r)\sum_{n=1}^{\infty} s_n r^n.$$

The right hand side tends to 0 as $r \to 1$. Therefore, $\sum_{n=1}^{\infty} c_n$ is Abel summable to 0.

(b) Suppose that $\sigma = 0$.

$$\sum_{n=1}^{N} c_n r^n = \sum_{n=1}^{N} (s_n - s_{n-1}) r^n = \sum_{n=1}^{N} \left[n\sigma_n - (n-1)\sigma_{n-1} - (n-1)\sigma_{n-1} + (n-2)\sigma_{n-2} \right] r^n$$

$$\cancel{= \sum_{n=1}^{N-2} n\sigma_n (r^2 - 2r + 1) + (N-1)\sigma_{N-1}(-2r}$$

$$= (r^2-2r+1)\sum_{n=1}^{N-2} n\sigma_n r^n + (-2r+1)(N-1)\sigma_{N-1} r^{N-1} + N\sigma_n r^N$$

Since there exists $M>0$. s.t. $|\sigma_n| < M$ for all $n \in \mathbb{N}^*$,

$$|n\sigma_n r^n| < M\left(\frac{1+r}{2}\right)^n \text{ for large enough } n.$$

and thus $\sum_{n=1}^{\infty} n\sigma_n r^n$ is convergent.

So letting $N\to\infty$, one has

$$\sum_{n=1}^{\infty} c_n r^n = (1-r)^2 \sum_{n=1}^{\infty} n\sigma_n r^n. \quad \square$$

$$\boxed{\text{Convergent} \Rightarrow \text{Cesàro summable} \Rightarrow \text{Abel summable}}$$

2.6.14. (a) If $\sum c_n$ is Cesàro summable to σ and $c_n = o(\frac{1}{n})$

then $\sum c_n$ converges to σ

(b) —— Abel summable

Proof. (a) Since $S_n = \sum_{k=1}^{n} c_k$, $\sigma_n = \frac{\sum^n S_k}{n}$, one has

$$S_n - \sigma_n = \frac{(n-1)c_n + \cdots + c_2}{n}.$$

It follows from $c_n = o(\frac{1}{n})$ that

$$\exists M>0 \text{. st. } |nc_n| < M \text{ for all } n\in\mathbb{N}^*$$

and

$$\exists N>0 \text{. st. for all } n>N, \ |nc_n| < \frac{\varepsilon}{2}.$$

Then for any $n > \max\{N, \frac{2NM}{\varepsilon}\}$,

$$|S_n - \sigma_n| \leq \frac{|(n-1)C_n| + \cdots + |NC_{N+1}| + |(N-1)C_N| + \cdots + C_2}{n}$$

$$\leq \frac{(n-N)\frac{\varepsilon}{2}}{n} + \frac{NM}{n} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon.$$

So $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sigma_n = \sigma$.

(b) $$\left| S_n - \sum_{k=1}^{\infty} C_k r^k \right| \leq \left| \sum_{k=1}^{n} C_k (1-r^k) \right| + \left| \sum_{k=N+1}^{\infty} C_k r^k \right|$$

$$:= \mathrm{I} + \mathrm{II}$$

$$|\mathrm{II}| \leq \frac{1}{n} \sum_{k=n+1}^{\infty} |kC_k| r^k \leq \frac{\varepsilon}{n} \quad \text{for all } n > N,$$

where N is defined as that in (a).

$$|\mathrm{I}| \leq (1-r) \sum_{k=1}^{n} |kC_k| \quad \text{since } (1-r^k) = (1-r)(1+\cdots+r^{k-1})$$

$$\leq (1-r)\, n M. \qquad\qquad \leq k(1-r)$$

Therefore, for any $\varepsilon > 0$, $\exists N$ defined as that in (a).

$$r = 1 - \frac{\varepsilon}{2(N+1)M}$$

$$\left| S_n - \sum_{k=1}^{\infty} C_k r^k \right| \leq \frac{\varepsilon}{N+1} + \frac{\varepsilon}{2} < \varepsilon.$$

Let $\varepsilon \to 0$, one has $N \to \infty$, $r \to 1$ and thus $\lim_{n\to\infty} S_n = \sigma$.

$\square$