Solution to Assignment 9

Ex 11. (p. 164) u is clearly continuous on $\{(x,t) : t > 0\}$ since u is infinitely differentiable there. It remains to show u is continuous on the x-axis, $\{(x,t) : t = 0\}$.

Note that we know already that (from good kernel argument)

$$u(x,t) \to f(x) \quad \text{as} \quad t \to 0, \text{ uniformly in } x.$$

Also, $f(x) = u(x,0)$ is continuous in x. From

$$u(x,t) - u(x_0,0) = (u(x,t) - u(x,0)) + (u(x,0) - u(x_0,0)),$$

it is easy to see $u(x,t)$ is continuous at the x-axis.

To see it vanishes at infinity, we note the following two estimates:

$$|u(x,t)| \le \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} |f(x-y)| dy \le \frac{c}{\sqrt{t}},$$

and

$$\begin{aligned} |u(x,t)| &\le \int_{|y| \le |x|/2} |f(x-y)| \mathcal{H}_t(y) dy + \int_{|y| > |x|/2} |f(x-y)| \mathcal{H}_t(y) dy \\ &\le \frac{C}{1+|x|^2} + Ct^{-1/2} e^{-cx^2/t}. \end{aligned}$$

Here, f is rapidly decreasing, so $|f(x-y)| \le C/(1+|x|^2)$ on $|y| \le |x|/2$; and $\mathcal{H}_t(y) \le Ct^{-1/2}e^{-cx^2/t}$ if $|y| > |x|/2$. To obtain it vanishes at infinity as $|x| + t \to \infty$, we note that if $|x| \le t$, then $t \to \infty$. We have

$$|u(x,t)| \le \frac{C}{\sqrt{t}} \to 0.$$

On the other hand, if $|x| > t$, then $|x| \to \infty$, we have

$$|u(x,t)| \le \frac{C}{1+|x|^2} + Ct^{-1/2} e^{-cx^2/t} \to 0.$$

15(a). **(p.165)** Let $f(x) = g(x)e^{-2\pi i x\alpha}$, where g is the function in Exercise 2. Then we have

$$\widehat{f}(\xi) = \widehat{g}(\xi + \alpha) = \left(\frac{\sin \pi(\xi + \alpha)}{\pi(\xi + \alpha)}\right)^2.$$

By the Poisson summation Formula,

$$1 = \sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) = \sum_{n=-\infty}^{\infty} \left(\frac{\sin(n\pi + \alpha\pi)}{\pi(n+\alpha)}\right)^2.$$

Hence,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\alpha)^2} = \frac{\pi^2}{(\sin \pi\alpha)^2}.$$

15(b). Since the equality is 1-periodic, it's sufficient to prove for the case $0 < \alpha < 1$. If $\alpha \neq \frac{1}{2}$, we have

$$\int_{\frac{1}{2}}^{\alpha} \frac{\pi^2}{(\sin \pi x)^2} dx = [(-\pi) \cot \pi x]_{\frac{1}{2}}^{\alpha} = -\frac{\pi}{\tan \pi\alpha}.$$

If we let $h_k(x) = \sum_{n=-k}^{k} \frac{1}{(n+x)^2}$,then $|h_k(x)| \leq \frac{\pi^2}{(\sin \pi x)^2}$ which is an integrable function. By the Lebesgue Dominated Convergence Theorem,

$$\lim_{k\to\infty} \int_{\frac{1}{2}}^{\alpha} h_k(x)dx = \int_{\frac{1}{2}}^{\alpha} \frac{\pi^2}{(\sin \pi x)^2} dx = -\frac{\pi}{\tan \pi\alpha}.$$

On the other hand,

$$\lim_{k\to\infty} \int_{\frac{1}{2}}^{\alpha} h_k(x)dx = \sum_{n=-\infty}^{\infty} \int_{\frac{1}{2}}^{\alpha} \frac{1}{(n+x)^2} dx = -\sum_{n=-\infty}^{\infty} \frac{1}{n+\alpha}.$$

So we have

$$\sum_{n=-\infty}^{\infty} \frac{1}{n+\alpha} = \frac{\pi}{\tan \pi\alpha}.$$

If $\alpha = \frac{1}{2}$, we can see easily that $\sum_{-N}^{N} \frac{1}{n+\frac{1}{2}} = \frac{1}{N+\frac{1}{2}}$. Hence, $\sum_{n=-\infty}^{\infty} \frac{1}{n+\alpha} = 0 = \lim_{\alpha\to 1/2} \frac{\pi}{\tan \pi\alpha}$. The formula continues to hold.

Ex 19. (p. 166) (a) Applying the Poisson summation formula to $f(x) = \frac{t}{\pi(x^2+t^2)}$ at $x = 0$ gives

$$\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{t}{t^2+n^2} = \sum_{n=-\infty}^{\infty}\widehat{f}(n) = \sum_{n=-\infty}^{\infty}e^{-2\pi t|n|}.$$

(b) By the Taylor series of $\frac{1}{1-x}$, one has for $0 < t < 1$, $n \neq 0$,

$$\begin{aligned}\frac{t}{t^2+n^2} &= \frac{t}{n^2}\frac{1}{1-\left(-\frac{t^2}{n^2}\right)} = \frac{t}{n^2}\sum_{l=0}^{\infty}\left(-\frac{t^2}{n^2}\right)^l \\ &= \sum_{m=1}^{\infty}(-1)^{m+1}\frac{1}{n^{2m}}t^{2m-1}.\end{aligned}$$

Note that this series converges uniformly for $t < 1$, and therefore

$$\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{t}{t^2+n^2} = \frac{1}{\pi t} + 2\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{t}{t^2+n^2} = \frac{1}{\pi t} + \frac{2}{\pi}\sum_{m=1}^{\infty}(-1)^{m+1}\zeta(2m)t^{2m-1}.$$

(For $t \geq 1$, the series in RHS does not converge since $1 \leq \zeta(2m) \leq \zeta(2)$ and $(-1)^{m+1}\zeta(2m)t^{2m-1}$ does not converge to 0 as m tends to infinity.)

Moreover,

$$\begin{aligned}\sum_{n=-\infty}^{\infty}e^{-2\pi t|n|} &= 1 + 2\sum_{n=1}^{\infty}\left(e^{-2\pi t}\right)^n \\ &= 1 + 2\left(\frac{1}{1-e^{-2\pi t}} - 1\right) = \frac{2}{1-e^{-2\pi t}} - 1.\end{aligned}$$

(c) By (a) and (b), one has

$$\frac{1}{\pi t} + \frac{2}{\pi}\sum_{m=1}^{\infty}(-1)^{m+1}\zeta(2m)t^{2m-1} = \frac{2}{1-e^{-2\pi t}} - 1.$$

Let $z = -2\pi t$, then the above equation can be written in z as follows:

$$\frac{z}{e^z-1} = 1 - \frac{z}{2} + \frac{2}{\pi}\sum_{m=1}^{\infty}(-1)^{m+1}\frac{2\zeta(2m)}{(2\pi)^2 m}z^{2m}.$$

Since the Taylor series of $\frac{z}{e^z-1}$ is unique, then it follows the given fact that

$$2\zeta(2m) = (-1)^{m+1}\frac{(2\pi)^2 m}{(2m)!}B_{2m}.$$

$\square$