Solution to Assignment 7

Ex 11. (p. 124) See Tutorial 8.

Ex 12. (p. 124) Let $\theta = 2\pi x$ and $\tau = 4\pi^2 t$. Then

$$\begin{aligned} u(\theta,\tau) &= \sum_{n=-\infty}^{\infty} a_n e^{-n^2\tau} e^{in\theta} \\ &= \sum_{n=-\infty}^{\infty} \Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iny}dy\Big) e^{-n^2\tau}e^{in\theta} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\Big(\sum_{n=-\infty}^{\infty} e^{-n^2\tau}e^{in(\theta-y)}\Big)dy \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)h_\tau(\theta-y)dy = (f*h_\tau)(\theta). \end{aligned}$$

□

Ex 1. (p. 161) (a) Treating f as a function on $[-L/2, L/2]$, the Fourier coefficients are

$$a_n(L) = \frac{1}{L}\int_{-L/2}^{L/2} f(x)e^{-2\pi i\frac{n\pi}{L}}\ dx.$$

By the fact that f is supported on $[-M, M]$ and $M < L/2$, the Fourier transform of f is given by $\widehat{f}(\xi) = \int_{-L/2}^{L/2} f(x)e^{-2\pi i\xi x}\ dx$ and hence

$$a_n(L) = \frac{1}{L}\widehat{f}(\frac{n}{L}).$$

As $\widehat{f}$ is of moderate decrease,

$$\sum_n |a_n(L)| \le \frac{1}{L}\sum_n \frac{C}{1+|n/L|^2} < \infty.$$

This implies that

$$f(x) = \sum_{n=-\infty}^{\infty} a_n(L)e^{2\pi i(n/L)x} = \delta\sum_{n=-\infty}^{\infty}\widehat{f}(n\delta)e^{2\pi i(n\delta)x},$$

where $\delta = 1/L$.

(b) For any $\epsilon > 0$, using the fact that F is of moderate decrease, we can find some large N so that

$$\Big|\int_{-\infty}^{\infty} F(x)\ dx - \int_{-N}^{N} F(x)\ dx\Big| < \frac{\epsilon}{3}$$

and

$$\Big|\sum_{n=-\infty}^{\infty}\delta F(\delta n)-\sum_{|n|\le N/\delta}\delta F(\delta n)\Big|\le\Big|\sum_{|n|>N/\delta}\delta F(\delta n)\Big|\le\sum_{|n|>N/\delta}\delta\frac{C}{|n\delta|^2}$$
$$\le\frac{C}{\delta}\int_{N/\delta}^{\infty}\frac{1}{x^2}\,dx=\frac{C}{\delta}\Big(-\frac{1}{x}\Big)\Big|_{N/\delta}^{\infty}=\frac{C}{N}<\frac{\epsilon}{3}.$$

Finally by the definition of Riemann integral on $[-N,N]$, we can choose sufficiently small δ so that

$$\Big|\int_{-N}^{N}F(x)\,dx-\sum_{|n|\le N/\delta}\delta F(\delta n)\Big|<\frac{\epsilon}{3}.$$

Combining all the above estimates, we obtain for δ sufficiently small,

$$\Big|\int_{-\infty}^{\infty}F(x)\,dx-\sum_{n=-\infty}^{\infty}\delta F(\delta n)\Big|<\epsilon.$$

This completes the proof.

(c) Let $F(\xi)=\widehat{f}(\xi)e^{2\pi ix\xi}$, then F is of moderate decrease. Using the results in (a) and (b), we obtain easily that

$$f(x)=\delta\sum_{n=-\infty}^{\infty}F(\delta n)=\int_{-\infty}^{\infty}F(\xi)\,d\xi=\int_{-\infty}^{\infty}\widehat{f}(\xi)e^{2\pi ix\xi}\,d\xi. \qquad \square$$