Solution to Assignment 4

12 (p.62) We need to prove

$$\frac{s_1 + s_2 + \cdots + s_n}{n} - s = \frac{(s_1 - s) + (s_2 - s) + \cdots + (s_n - s)}{n} \to 0$$

as $n \to \infty$. By replacing s_n with $s_n - s$, it suffices to prove the case for $s = 0$. Now, given any $\epsilon > 0$, we note that $s_n \to 0$ and therefore there exists $N \in \mathbb{N}$ such that $|s_n| < \epsilon$ for all $n > N$. Then we have

$$\begin{aligned}\left|\frac{s_1 + s_2 + \cdots + s_n}{n}\right| &= \left|\frac{s_1 + \cdots + s_N + s_{N+1} + \cdots + s_n}{n}\right| \\ &\le \frac{|s_1 + \cdots + s_N|}{n} + \frac{1}{n}(|s_{N+1}| + \cdots + |s_n|) \\ &\le \frac{|s_1 + \cdots + s_N|}{n} + (\frac{n-N}{n})\epsilon \\ &< \frac{|s_1 + \cdots + s_N|}{n} + \epsilon.\end{aligned}$$

Since N is fixed and $|s_1 + \cdots + s_N|$ is a finite number, we can choose an integer $N_1 > N$ such that $\frac{|s_1 + \cdots + s_N|}{n} < \epsilon$. Hence, whenever $n > N_1$.

$$\left|\frac{s_1 + s_2 + \cdots + s_n}{n}\right| < 2\epsilon.$$

Thus $\sum c_n$ is Cesàro summable to s.

13(a) (p.62) By letting $c_1' = c_1 - s$, $c_n' = c_n$ for $n \ge 2$, we see that the series $\sum c_n$ is Abel summable to s if and only if c_n' is Abel summable to 0. Hence it suffices to consider $s = 0$. Let $s_0 = 0$ and $s_n = c_1 + ... + c_n$, then

$$\sum_{n=1}^{N} c_n r^n = \sum_{n=1}^{N} (s_n - s_{n-1}) r^n = \sum_{n=1}^{N} s_n r^n - r \sum_{n=1}^{N-1} s_n r^n = (1-r) \sum_{n=1}^{N} s_n r^n + s_N r^{N+1}.$$

Since $s_N r^{N+1} \to 0$ as $N \to \infty$, thus

$$\sum_{n=1}^{\infty} c_n r^n = (1-r) \sum_{n=1}^{\infty} s_n r^n.$$

For any $\epsilon > 0$, by noting that $s_n \to 0$, we can therefore find $N_0 \in \mathbb{N}$, such that $|s_n| < \epsilon$ for $n > N_0$. Moreover, $s_n \to 0$ implies $|s_n| \le M$, we can find $\delta > 0$, such that $(1-r)MN_0 < \epsilon$ whenever $1-\delta < r < 1$. Then we have

$$\begin{aligned}\left|(1-r)\sum_{n=1}^{\infty} s_n r^n\right| \leq& \left|(1-r)\sum_{n=1}^{N_0} s_n r^n\right| + (1-r)\sum_{n=N_0+1}^{\infty} |s_n| r^n \\ \leq&(1-r)MN_0 + \epsilon \sum_{n=N_0+1}^{\infty} r^n \\ =&\epsilon + \epsilon = 2\epsilon.\end{aligned}$$

This means $\lim_{r\to 1}(1-r)\sum_{n=1}^{\infty} s_n r^n = 0$. Hence $\sum c_n$ is Abel summable.

13(b) Let $c_n = (-1)^n$, then $\sum_1^{\infty}(-1)^n$ does not converge. However,

$$\lim_{r\to 1}\sum_{n=1}^{\infty}(-1)^n r^n = \lim_{r\to 1}\frac{-r}{1+r} = -\frac{1}{2}.$$

13(c) Again, we just need to consider $\sigma = 0$. Recall that $\sigma_n = \frac{s_1+...+s_n}{n}$, from 13(a), we obtain (using the same identity with c_n replaced by s_n)

$$\sum_{n=1}^{\infty} c_n r^n = (1-r)\sum_{n=1}^{\infty} s_n r^n = (1-r)^2 \sum_{n=1}^{\infty} n\sigma_n r^n.$$

We also further recall the identity $\sum_{n=1}^{\infty} nr^n = r\sum_{n=1}^{\infty}\frac{dr^n}{dr} == r\frac{d}{dr}(\frac{r}{1-r}) = \frac{r}{(1-r)^2}$. Now, for any $\epsilon > 0$, there exists N_0 such that for all $n \geq N_0$, $|\sigma_n| < \epsilon$. Hence,

$$(1-r)^2\left|\sum_{n=N_0+1}^{\infty} n\sigma_n r^n\right| \leq (1-r)^2\left(\sum_{n=1}^{\infty} nr^n\right)\epsilon < \epsilon r < \epsilon.$$

Moreover, $|\sigma_n| \leq M$ for all n. We then take $\delta = \sqrt{\frac{\epsilon}{(\sum_{n=1}^{N_0} n)M}} > 0$ such that whenever $1-\delta < r < 1$,

$$\left|(1-r)^2(\sum_{n=1}^{N_0} n\sigma_n r^n)\right| \leq (1-r)^2(\sum_{n=1}^{N_0} n)M < \epsilon.$$

Combining this, we obtain whenever $1-\delta < r < 1$,

$$\left|\sum_{n=1}^{\infty} c_n r^n\right| = \left|(1-r)^2(\sum_{n=1}^{N_0} n\sigma_n r^n) + (1-r)^2\sum_{n=N_0+1}^{\infty} n\sigma_n r^n\right| < 2\epsilon.$$

This completes the proof.

13(d) Note that if c_n is Cesàro summable (i.e. $\sigma_n = \frac{s_1+..+s_n}{n} \to \sigma$), then

$$\frac{s_n}{n} = \frac{n\sigma_n - (n-1)\sigma_{n-1}}{n} = \sigma_n - \frac{n-1}{n}\sigma_{n-1} \longrightarrow 0.$$

Hence,

$$\frac{c_n}{n} = \frac{s_n - s_{n-1}}{n} = \frac{s_n}{n} - \frac{n-1}{n}\frac{s_{n-1}}{n-1} \longrightarrow 0.$$

If $c_n = (-1)^{n-1}n$, then $\frac{c_n}{n} = (-1)^{n-1}$, which does not converge. Hence, c_n is not Cesàro summable. However, $\sum_{n=1}^{\infty}(-1)^{n-1}nr^n = \frac{r}{(1+r)^2}$, so

$$\lim_{r\to 1}\sum_{n=1}^{\infty}(-1)^{n-1}nr^n = \frac{1}{4}.$$

Hence, it is Abel summable.

Ex 6 (p. 89). Assume that $\{a_k\}$ is the coefficient of some Riemann integrable function f, i.e. $f(x) \sim \sum_{k=1}^{\infty}\frac{e^{ikx}}{k}$. Consider $A_r(f)(0)$,

$$A_r(f)(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)P_r(-\theta)d\theta.$$

$P_r(\theta)$ is an even function on θ, and since $1-2r\cos\theta+r^2 = (1-r\cos\theta)^2+r^2\sin^2\theta > 0$ for $r \in [0,1)$, so $P_r(\theta) = P_r(-\theta) = \frac{1-r^2}{1-2r\cos\theta+r^2} > 0$. We now have

$$\begin{aligned} |A_r(f)(0)| \leq & \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta)|P_r(-\theta)d\theta \\ \leq & \frac{1}{2\pi}\sup_{\theta}|f(\theta)|\int_{-\pi}^{\pi}P_r(-\theta)d\theta \\ = & \sup_{\theta}|f(\theta)| < \infty. \end{aligned}$$

On the other hand, $\lim_{r\to 1} A_r(f)(0) = \lim_{r\to 1}\sum_{k=1}^{\infty}\frac{r^k}{k} = \infty$. Therefore, there's no function with $\{a_k\}$ as its coefficient.

Note that $\lim_{r\to 1}\sum_{k=1}^{\infty}\frac{r^k}{k} = \infty$ *because for all* $M > 0$, *we can choose* N *such that* $\sum_{k=1}^{N}\frac{1}{k} > 2M$. *Then we choose* r *so close to 1 that* $r^N \geq 1/2$, *then*

$$\sum_{k=1}^{\infty}\frac{r^k}{k} \geq \sum_{k=1}^{N}\frac{r^k}{k} \geq \frac{1}{2}\sum_{k=1}^{N}\frac{1}{k} \geq M.$$

Ex 8a (p. 89). $\widehat{f}(n)$ is the same as that of Exercise 6 in Chapter 2. Using the Parseval's identity:

$$(\frac{1}{2\pi})^2 + 2\sum_{n=0}^{\infty}\left(\frac{-2}{(2n+1)^2\pi}\right)^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta)|^2 d\theta = \frac{\pi^2}{3}.$$

This implies

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} = \frac{\pi^4}{96}.$$

Also, $\sum_{n=1}^{\infty}\frac{1}{n^4} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} + \frac{1}{2^4}\sum_{n=1}^{\infty}\frac{1}{n^4}$. Hence,

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

Ex 8b. We have computed $\widehat{f}(n)$ in Exercise 4 in Chapter 2. Using the same method as (a), we have

$$2 \cdot \frac{16}{\pi^2} \cdot \sum_{k>0, k \text{ odd}} \frac{1}{k^6} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta = \frac{\pi^4}{30}.$$

Hence, $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^6} = \frac{\pi^6}{960}$ follows. As

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^6} + \frac{1}{2^6} \sum_{n=1}^{\infty} \frac{1}{n^6},$$

we have $\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$.

Ex 9. (p.90) We note that

$$\begin{aligned}
\widehat{f}(n) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{\pi}{\sin \pi\alpha} e^{i(\pi - x)\alpha} e^{-inx} dx \\
&= \frac{1}{2 \sin \pi\alpha} \int_0^{2\pi} e^{i\pi\alpha} e^{-i(n+\alpha)x} dx \\
&= \frac{e^{i\pi\alpha}}{2 \sin \pi\alpha} \left(-\frac{1}{i(n+\alpha)} e^{-i(n+\alpha)x} |_0^{2\pi} \right) \\
&= \frac{1}{n+\alpha}.
\end{aligned}$$

Hence the Fourier series of f is $\sum_{-\infty}^{\infty} \frac{e^{inx}}{n+\alpha}$. By the Parseval's identity,

$$\sum_{-\infty}^{\infty} \frac{1}{(n+\alpha)^2} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\pi}{\sin \pi\alpha} e^{i(\pi - x)\alpha} dx = \frac{\pi^2}{\sin^2 \pi\alpha}.$$