Solution to Assignment 2

4(a) (p.59)

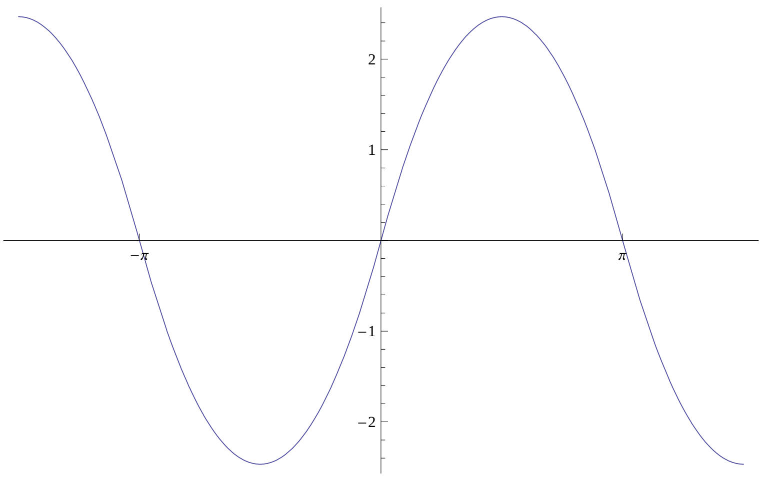

FIGURE 1. $f(\theta) = \theta(\pi - \theta)$, with odd extension

4(b). If $n = 0$, it is clear that $\widehat{f}(0) = 0$. If $n \neq 0$, we calculate the Fourier coefficients as follows:

$$\begin{aligned}\widehat{f}(n) =& \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta \\ =& -\frac{i}{2\pi}\int_{-\pi}^{\pi} \theta(\pi-\theta)\sin n\theta d\theta \\ =& \frac{-i}{\pi}\int_{0}^{\pi} \theta(\pi-\theta)(\sin n\theta)d\theta,\end{aligned}$$

where we have used f is an odd function and $e^{i\theta} = \cos\theta + i\sin\theta$. Using integration by part, we have

$$\int_0^\pi \theta \sin n\theta d\theta = -\frac{\pi(-1)^n}{n}, \quad \int_0^\pi \theta^2 \sin n\theta d\theta = -\frac{\pi^2((-1)^n)}{n} + \frac{2}{n^3}((-1)^n - 1).$$

Hence,

$$\widehat{f}(n) = \frac{-2i}{n^3\pi}(1-(-1)^n) = \begin{cases} \frac{-4i}{n^3\pi}, & \text{if } n \text{ is odd;} \\ 0, & \text{if } n \text{ is even.}\end{cases}$$

This shows the Fourier series of f is given by

$$\sum_{n=odd, n\in\mathbb{Z}} \widehat{f}(n)e^{in\theta} = \frac{8}{\pi}\sum_{n\geq 1, n=odd} \frac{\sin n\theta}{n^3}.$$

As $\sum |\widehat{f}(n)| \leq C \sum_n \frac{1}{n^3} < \infty$, for some constant $C > 0$, the Fourier series is equal to f (Corollary 2.3 of the book).

$$f(\theta) = \frac{8}{\pi} \sum_{n \geq 1, n = odd} \frac{\sin n\theta}{n^3}.$$

6(a) (p. 60)

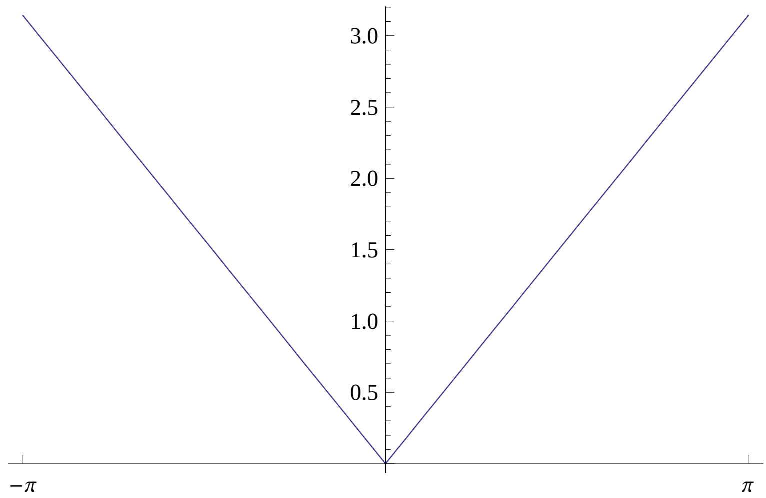

FIGURE 2. $f(\theta) = |\theta|$

6(b). If $n = 0$, $\widehat{f}(0) = \frac{1}{\pi} \int_0^\pi \theta d\theta = \frac{\pi}{2}$. If $n \neq 0$, using f is even

$$\begin{aligned} \widehat{f}(n) =& \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta \\ =& \frac{1}{2\pi} \int_{-\pi}^{\pi} |\theta| \cos n\theta d\theta \\ =& \frac{1}{\pi} \int_0^\pi \theta \cos n\theta d\theta. \\ =& \frac{-1 + (-1)^n}{\pi n^2}. \end{aligned}$$

6(c). Note that $\widehat{f}(n) e^{in\theta} + \widehat{f}(-n) e^{-in\theta} = \begin{cases} \frac{-4}{\pi n^2} \cos n\theta, & \text{if } n \text{ is odd ;} \\ \frac{\pi}{2}, & \text{if } n = 0; \\ 0, & \text{if } n \text{ is even.} \end{cases}$ we have,

$$f(\theta) \sim \frac{\pi}{2} + \sum_{n \geq 1,\ n = odd} \frac{-4}{\pi n^2} \cos n\theta.$$

6(d). As $\sum |\widehat{f}(n)| \leq C \sum_n \frac{1}{n^2} < \infty$, for some constant $C > 0$, the Fourier series is equal to f (Corollary 2.3 of the book).

$$f(\theta) = \frac{\pi}{2} + \sum_{n\geq 1, n=odd} \frac{-4}{\pi n^2} \cos n\theta.$$

Taking $\theta = 0$, we have

$$0 = f(0) = \frac{\pi}{2} - \sum_{n\geq 1, n=odd} \frac{4}{\pi n^2}.$$

This implies that

$$\sum_{n\geq 1, n=odd} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

Finally,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n\geq 1, n=odd} \frac{1}{n^2} + \sum_{n\geq 1, n=even} \frac{1}{n^2} = \frac{\pi^2}{8} + \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

This implies

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

10 (p.61) As $f \in C^k$ and $f(2\pi) = f(0)$, we have by successive integration by part (for $n \neq 0$),

$$\begin{aligned}
\widehat{f}(n) =& \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta \\
=& \frac{1}{2\pi i n} \int_{-\pi}^{\pi} f'(\theta) e^{-in\theta} d\theta \\
=& \vdots \\
=& \frac{1}{2\pi (in)^k} \int_{-\pi}^{\pi} f^{(k)}(\theta) e^{-in\theta} d\theta.
\end{aligned}$$

Note that $f \in C^k$ means $f^{(k)}$ is continuous on $\mathbb{T}$. This means there exists $M > 0$ such that $|f^{(k)}(x)| \leq M$ for all x. Hence,

$$|\widehat{f}(n)| \leq |\frac{1}{2\pi (in)^k}| \cdot M \leq \frac{C}{|n|^k},$$

where C is some constant independent of n. This shows that $\widehat{f}(n) = O(\frac{1}{|n|^k})$.