Solution to Assignment 1

1. If $n=0$, $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx = 1$. If $n \neq 0$, then $e^{-in\pi} = e^{-in\pi+2in\pi} = e^{in\pi}$ and hence we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}dx = \frac{1}{2\pi in}[e^{inx}]_{-\pi}^{\pi} = \frac{e^{in\pi}-e^{-in\pi}}{2\pi in} = 0.$$

For the second part, we note that $\cos x = \frac{e^{ix}+e^{-ix}}{2}$ and $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$, so

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx \cdot \cos mx dx = \frac{1}{4\pi}\int_{-\pi}^{\pi} e^{i(n+m)x} + e^{-i(n+m)x} + e^{i(n-m)x} + e^{-i(n-m)x} dx$$

As $n, m \geq 1$, $n+m \neq 0$. By the above results,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx \cdot \cos mx dx = \begin{cases} 0, & \text{if } n \neq m; \\ 1, & \text{if } n = m. \end{cases}$$

Similarly,

$$\begin{aligned} \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx \cdot \sin mx dx =& \frac{-1}{4\pi}\int_{-\pi}^{\pi} e^{i(n+m)x} + e^{-i(n+m)x} - e^{i(n-m)x} - e^{-i(n-m)x} dx \\ =& \begin{cases} 0, & \text{if } n \neq m; \\ 1, & \text{if } n = m. \end{cases} \end{aligned}$$

$$\begin{aligned} \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx \cdot \cos mx dx =& \frac{1}{4\pi i}\int_{-\pi}^{\pi} e^{i(n+m)x} - e^{-i(n+m)x} + e^{i(n-m)x} - e^{-i(n-m)x} dx \\ =& 0. \end{aligned}$$

2(a). In polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$. We see that $U(r,\theta) = u(r\cos\theta, r\sin\theta)$, and hence,

$$\frac{\partial U}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta, \quad \frac{\partial U}{\partial \theta} = -r\frac{\partial u}{\partial x}\sin\theta + r\frac{\partial u}{\partial y}\cos\theta.$$

Differentiate one more time, we get

$$\begin{aligned} \frac{\partial^2 U}{\partial r^2} =& \frac{\partial \frac{\partial u}{\partial x}\cos\theta}{\partial r} + \frac{\partial \frac{\partial u}{\partial y}\sin\theta}{\partial r} \\ =& \frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial y \partial x}\sin\theta\cos\theta + \frac{\partial^2 u}{\partial x \partial y}\cos\theta\sin\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta \\ =& \frac{\partial^2 u}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 u}{\partial x \partial y}\sin\theta\cos\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta, \end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 U}{\partial\theta^2} =& \frac{\partial(-r\frac{\partial u}{\partial x}\sin\theta)}{\partial\theta} + \frac{\partial(r\frac{\partial u}{\partial y}\cos\theta)}{\partial\theta}\\ =&\frac{\partial^2 u}{\partial x^2}r^2\sin^2\theta + \frac{\partial^2 u}{\partial y^2}r^2\cos^2\theta - 2\frac{\partial^2 u}{\partial x\partial y}r^2\sin\theta\cos\theta - \frac{\partial u}{\partial x}r\cos\theta - \frac{\partial u}{\partial y}r\sin\theta.\end{aligned}$$

Hence, combining all the above formulas, it is easily seen that

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial\theta^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Remark. *For a more natural derivation, one should show that*

$$\begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{bmatrix} = \begin{bmatrix}\cos\theta & -\frac{\sin\theta}{r}\\ \sin\theta & \frac{\cos\theta}{r}\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial r}\\ \frac{\partial}{\partial\theta}\end{bmatrix}.$$

Then compute $\frac{\partial^2}{\partial x^2}$, $\frac{\partial^2}{\partial y^2}$ to get the result. For detail, please refer to the PDE textbook: "Partial differential equations, An introduction, Walter Strauss".

2(b). Let $U(r,\theta) = F(r)G(\theta)$ and put it into the polar Laplace equation.

$$\frac{r^2F''(r) + rF'(r)}{F(r)} = -\frac{G''(\theta)}{G(\theta)}. \tag{0.1}$$

The left hand side of (0.1) is a function of r while the right hand side of (0.1) is a function of θ, so the expression must be a constant α. Hence, we obtain $G''+\alpha G = 0$. From elementary ODE, We have three cases

$$\begin{cases} G(\theta) = A_\alpha\cos(\sqrt{\alpha}\theta) + B_\alpha\sin(\sqrt{\alpha}\theta), & \text{if } \alpha > 0;\\ G(\theta) = A_\alpha\theta + B_\alpha, & \text{if } \alpha = 0;\\ G(\theta) = A_\alpha\cosh(\sqrt{-\alpha}\theta) + B_\alpha\sinh(\sqrt{-\alpha}\theta), & \text{if } \alpha < 0.\end{cases}$$

Note that G must be 2π periodic, so $\alpha = n^2$, where $n = 0, 1, 2....$

(i)If $n = 0$, then $G(\theta) = B_0$. We have from (0.1)

$$r^2F''(r) + rF'(r) = 0.$$

Solving it using separation of variables, $F(r) = C\ln r + D$. However, $\ln r$ is unbounded when r goes to zero but the solution is bounded in the origin, hence we have

$$C = 0 \text{ and } F(r) = D$$

. The solution is of desired form.

(ii) If $n > 0$, then $\alpha = n^2$ and $G(\theta) = A_n\cos(n\theta) + B\sin(n\theta)$. Putting back to (0.1), we have

$$r^2F''(r) + rF'(r) - n^2F(r) = 0. \tag{0.2}$$

For this equation, we try $F(r) = r^k$ for some k. Then

$$(k(k-1) + k - n^2)r^k = 0.$$

As this needs to be true for all r, we must have $k^2 - n^2 = 0$ and hence $k = \pm n$. The ODE in (0.2) is of 2^{nd} order and linear, so the solution space is of dimension 2. Moreover, the solutions r^n and r^{-n} are linearly independent, this implies all the solutions to (0.2) are given by

$$F(r) = C_n r^n + D_n r^{-n}.$$

Since the solution is bounded in the origin, we have

$$D_n = 0 \text{ and } F(r) = C_n r^n$$

. Hence, the solution is given by

$$U(r, \theta) = F(r)G(\theta) = r^n (A_n \cos n\theta + B_n \sin n\theta).$$