THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH5022 Theory of Partial Differential Equations, 2nd Term 2016-17

Homework 3 (Due: April 20). Give answers to all questions and hand in your solutions directly to me in class on or before the due date.

Q1. In the lecture we showed that for a bounded domain in $\mathbb{R}^n$, the linear operator

$$D^2\Delta^{-1} : L^p(\Omega) \to L^p(\Omega)$$

is bounded for $1 < p < \infty$, where Δ^{-1} is defined in terms of the Newton potential as

$$\Delta^{-1} f = \int_\Omega \phi(x-y) f(y)\, dy,$$

with ϕ being the fundamental solution to Δ.

(a) Find an example for failure of the case $p = 1$.
(b) Find an example for failure of the case $p = \infty$.

Q2. Slightly modify the proof for $D^2\Delta^{-1}$ to show the following result: Let $K \in L^2(\mathbb{R}^n)$. Suppose:

(a) The Fourier transform of K is essentially bounded:

$$\sup_{\xi \in \mathbb{R}^n} |\hat{K}(\xi)| \leq B.$$

(b) K is of class of C^1 outside the origin: $K \in C^1(\mathbb{R}^n \setminus \{0\})$, and

$$|\nabla K(x)| \leq \frac{B}{|x|^{n+1}}.$$

For $f \in L^1 \cap L^p$, set

$$(Tf)(x) = \int_{\mathbb{R}^n} K(x-y) f(y)\, dy.$$

Then there exists a constant A such that

$$\|Tf\|_{L^p(\mathbb{R}^n)} \leq A\|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty.$$

One can thus extend T to all of L^p by continuity. Here, the constant A depends only on p, B and the dimension n. In particular, it does not depend on the L^2 norm of K.

Q3. Let $u \in C^2(\bar{\Omega})$, $u = 0$ on $\partial\Omega \in C^1$. Prove the interpolation inequality: For each $\epsilon > 0$,

$$\int_\Omega |Du|^2 \leq \epsilon \int_\Omega |\Delta u|^2 + \frac{1}{4\epsilon} \int_\Omega u^2.$$