THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH5022 Theory of Partial Differential Equations, 2nd Term 2016-17

Homework 2 (Due: April 6th). Give answers to all questions and hand in your solutions directly to me in class on or before the due date.

Q1. Let $u \in W^{1,1}(\Omega)$, and suppose that there are constants $M > 0$, $0 < \alpha \leq 1$ such that

$$\int_{B_R} |Du| \leq MR^{n-1+\alpha} \quad \text{for any } B_R \subset \Omega.$$

Then $u \in C^\alpha(\Omega)$, and for any $B_R \subset \Omega$

$$\operatorname*{Osc}_{B_R} u \leq CMR^\alpha,$$

where $C = C(n, \alpha) > 0$.

Hint: The result is due to Morrey. Compare it to Theorem 3.1 on page 48 of the textbook.

Q2. Modify slightly the proof of Lemma 3.7 (Page 53 of the textbook) to show: Let f be a non-negative integrable function on $\mathbb{R}^n$, and let α be a positive constant. Then there exists a decomposition of $\mathbb{R}^n$ so that

(i) $\mathbb{R}^n = F \cup \Omega$, $F \cap \Omega = \emptyset$.
(ii) $f(x) \leq \alpha$ a.e. on F.
(iii) Ω is the union of cubes, $\Omega = \cup_k Q_k$, where interiors are disjoint, and so that for each Q_k,

$$\alpha < \frac{1}{|Q_k|} \int_{Q_k} f \leq 2^n \alpha.$$

Hint: The result is a fundamental lemma by Calderon and Zygmund.

Q3. Let u be a weak solution to $-\Delta u = f$ in Ω, i.e.,

$$\int_\Omega D_i u D_i \varphi = \int_\Omega f\varphi \quad \text{for any } \varphi \in H_0^1(\Omega).$$

Find an example showing that $f \in C(\bar{\Omega})$, $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$ for any $\alpha \in (0,1)$ while D^2u is not continuous in Ω. What if $f \in C^\alpha(\bar{\Omega})$ and Ω has a smooth boundary? Justify your answer in detail.

Q4. Complete the proof of Theorem 4.13 (page 83 of the textbook) in detail.

Q5. On page 90 of the textbook, it is written that the estimate on line 6:

$$\int |Dw|^2\eta^2 \leq \frac{C}{(1-\gamma)^\alpha} \int w^2(|D\eta|^2 + \eta^2)$$

can be derived from the estimate on line 4:

$$\int_{B_1} |D\bar{u}|^2 \bar{u}^{-\beta-1}\eta^2 \leq C \left\{ \frac{1}{\beta^2} \int_{B_1} |D\eta|^2 \bar{u}^{1-\beta} + \frac{1}{\beta} \int_{B_1} \frac{|f|}{k} \eta^2 \bar{u}^{1-\beta} \right\},$$

where $k = \|f\|_{L^q} > 0$, but the details of the proof is omitted. Give reasons to fill the proof.

Hint: Apply Hölder with $\frac{1}{q} + \frac{q-1}{q} = 1$ to $\int_{B_1} \frac{|f|}{k}\eta^2 w^2$, interpolation to control $\|\eta w\|_{L^{2q/(q-1)}}$ by norms in L^{2^*} and L^2, and then conclude the proof with Young's inequality with $\epsilon > 0$, Sobolev inequality $\dot{H}^1 \subset L^{2^*}$.