Answer to a Question Posed in class March 10:

Claim: Let y be recurrent, then $\lim_{n\to\infty} \frac{N_n(y)}{n} \overset{\text{prob.}}{=} \frac{1_{\{T_y<\infty\}}}{m_y}$.

Proof. $\{T_y<\infty\} = \{1 \le T_y < \infty\} = \bigcup_{k=1}^{\infty} \{T_y = k\}$

Consider the limit in two cases:

$T_y = k$ for $k \in \{1, 2, \cdots\}$ and $T_y = \infty$

<u>Case 1.</u> $T_y = \infty$, to show: $\lim_{n\to\infty} \frac{N_n(y)}{n} \overset{\text{prob.}}{=} 0$. $\left(\because 1_{\{T_y<\infty\}} = 0 \text{ if } T_y = \infty \right)$

Indeed, recall "$T_y = \infty$" means

$$X_m \neq y, \quad \forall m \geq 1$$

$$\therefore N_n(y) = \sum_{m=1}^{n} 1_y(X_m) = 0, \quad \forall n \geq 1$$

$$\therefore \lim_{n\to\infty} \frac{N_n(y)}{n} = \lim_{n\to\infty} \frac{0}{n} = \lim_{n\to\infty} 0 = 0.$$

<u>Case 2</u>. $T_y = k$ for a given finite $k \in \{1, 2, \cdots\}$.

to show: $\displaystyle \lim_{n\to\infty} \frac{N_n(y)}{n} = \frac{1}{m_y}$. $\left(\because \mathbb{1}_{\{T_y<\infty\}} = 1 \text{ if } T_y = k < \infty \right)$

Indeed, recall

$$k = T_y = \min\{ m \geq 1 : X_m = y \}$$

implying

$$X_1 \neq y, \cdots, X_{k-1} \neq y, X_k = y \qquad (*)$$

We then write: for any $n > k$,

$$\frac{N_n(y)}{n} = \frac{\sum_{m=1}^{n} 1_y(X_m)}{n}$$

$$= \frac{\sum_{m=1}^{k} 1_y(X_m) + \sum_{m=k+1}^{n} 1_y(X_m)}{n}$$

$$= \frac{1}{n} + \frac{\sum_{m=k+1}^{n} 1_y(X_m)}{n} \qquad (**)$$

$$\left(\because \sum_{m=1}^{k} 1_y(X_m) = 1 \text{ by } (*) \right)$$

Note :

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

and

$$\lim_{n\to\infty} \frac{\sum_{m=k+1}^{n} 1_y(X_m)}{n} = \lim_{n\to\infty} \frac{\sum_{m=k+1}^{n} 1_y(X_m)}{n-k} \cdot \frac{n-k}{n}$$

$$= \lim_{n\to\infty} \frac{\sum_{m=k+1}^{n} 1_y(X_m)}{n-k}$$

Now, given $X_k = y \in S_R$ at time k, we may restrict our chain over some C_i that is an irreducible closed set of recurrent states in the state decomposition

$$S_R = \bigcup_{i=1}^{r} C_i$$

(disjoint union)

As such, we have: $X_m \in C_i, \quad \forall\, m \geq k$

Due to the Markovian property, we may regard the chain starting at time k, then by the thm in the COURSE NOTE 10,

$$\lim_{n\to\infty} \frac{\sum_{m=k+1}^{n} \mathbb{1}_y(X_m)}{n-k}$$

$$= \lim_{n\to\infty} \frac{\sum_{m=1}^{n-k} \mathbb{1}_y(\tilde{X}_m)}{n-k} \qquad \left(\tilde{X}_m \overset{\text{def.}}{=} X_{m+k} \quad m = 0, 1, \cdots \right)$$

$$\overset{\text{prob.}}{=} \frac{1}{m_y}$$

Plugging those back to $(**)$,

$$\lim_{n\to\infty} \frac{N_n(y)}{n} \overset{\text{prob.}}{=\!=} \lim_{n\to\infty} \frac{1}{n} + \lim_{n\to\infty} \frac{\sum_{m=k+1}^{n} 1_y(X_m)}{n}$$

$$\overset{\text{prob.}}{=\!=} 0 + \frac{1}{m_y}$$

$$= \frac{1}{m_y}.$$

Combing two cases above gives the proof of the desired result $\lim_{n\to\infty} \frac{N_n(y)}{n} = \frac{1_{\{T_y<\infty\}}}{m_y}$, $\forall\, y \in S_R$ for a general Markov chain. ###