The Chinese University of Hong Kong
MATH3093 Fourier Analysis 2021-2022
Mid-term Examination

March 16, 2022 19:30pm–21:30pm

Instructions

- This is an open-book examination. You may **ONLY** refer to printed/written materials during the examination. Assessing information on the internet is not allowed.
- You are allowed to use a calculator in the approved list during the examination.
- You shall take the examination in isolation and shall not communicate with any person during the examination other than the course teacher(s) concerned. Please kindly be reminded of the following regulations enforced by the university:

  Honesty in Academic Work: *The Chinese University of Hong Kong places very high importance on honesty in academic work submitted by students, and adopts a policy of zero tolerance on cheating and plagiarism. Any related offence will lead to disciplinary action including termination of studies at the University.*

- There are a total of 100 points.
- Answer **ALL** questions.
- Show all steps clearly in your working. **NO** point will be given if suffcient justification is not provided.
- Only handwritten answers on papers or electronic devices will be accepted. Typed answer will **NOT** be accepted.
- Please follow the instruction of submission below:
  1. Write your answers on papers or electronic devices. Only handwritten answers will be accepted and typed answer will **NOT** be accepted.
  2. Scan or take photos of your work (if you write on papers).
  3. Combine your work into a single pdf file.
  4. Name the pdf file by your student id (e.g. 1155123456.pdf).
  5. You must upload the pdf file to Blackboard before 21:30pm, 16 March, 2022. Mark deduction will be made for late submission.
  6. Please check your file carefully to make sure no missing pages.

1 (10pts). For any real-valued Riemann integrable function f on the circle, there are two ways expressing its Fourier series

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

and

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

Relate $c_n (n \in \mathbb{Z})$ to $a_n (n \geq 0)$ and $b_n (n \geq 1)$.

2 (25pts). Let $f(x) = x$ on $[-\pi, \pi]$.

(a) Calculate the Fourier series of f on $[-\pi, \pi]$.

(b) Find a trigonometric polynomial P of degree 2 (that is, P is of the form $\sum_{|n|\leq 2} c_n e^{inx}$) such that

$$\|f - P\|^2 := \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - P(x)|^2 \, dx$$

is the smallest. Furthermore, find the value of $\|f - P\|$.

3 (25 pts).

(a) Calculate the Fourier series of the 2π-periodic function defined on $[-\pi, \pi]$ by $f(x) = e^x$.

(b) Calculate $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{n^2+1}$.

4 (10pts) Suppose that f is a 2π-periodic function which belongs to the class C^k ($k \in \mathbb{N}$). Show that there exists a constant $C > 0$ such that

$$|\widehat{f}(n)| \leq \frac{C}{|n|^k}, \quad \forall\, n \neq 0.$$

——————(Please Turn Over)——————

5 (10pts). Show that the trigonometric series

$$\sum_{n=2}^{\infty} \frac{1}{\log n} e^{inx}$$

is not the Fourier series of a Riemann integrable function on $[-\pi, \pi]$.

6 (10pts) Let $f(x)$ be the function on $[-\pi, \pi]$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, \pi], \\ 0 & \text{otherwise} . \end{cases}$$

It is known that the Fourier series of f is given by

$$f(x) \sim \frac{1}{2} + \sum_{k=-\infty}^{\infty} \frac{1}{\pi i(2k+1)} e^{i(2k+1)x}.$$

Show that if $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ is the Fourier series of a Riemann integrable function on the circle, then so is $\displaystyle\sum_{n=-\infty}^{\infty} \frac{c_{2n+1}}{2n+1} e^{i(2n+1)x}$.

7 (10pts) Let f and g be 2π-periodic functions on $\mathbb{R}$ so that both are Riemann integrable on the circle. Prove that

$$\lim_{n \to +\infty} \frac{1}{2\pi} \int_0^{2\pi} f(x) g(nx) dx = \widehat{f}(0) \widehat{g}(0).$$

——— End ———