Q2 $f(x) = x$ on $[-\pi, \pi]$.

$$f(x) \sim \sum_{n \neq 0} \frac{(-1)^n}{n} i e^{inx}.$$

(b) Find a trigonometric polynomial of degree 2 s.t. $\|f - P\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - P(x)|^2 dx$ is the smallest and compute $\|f - P\|$.

$$P(x) = -ie^{ix} + \frac{1}{2} i e^{i2x} + ie^{-ix} - \frac{1}{2} e^{-i2x}.$$

$$\|f - P\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - P(x)|^2 dx$$

$$= \sum_{|n| > 2} \left| \frac{(-1)^n}{n} i \right|^2$$

$$= \sum_{|n| > 2} \frac{1}{n^2}$$

$$= 2 \sum_{n=3}^{\infty} \frac{1}{n^2}$$

$$= 2\left(\frac{\pi^2}{6} - 1 - \left(\frac{1}{2}\right)^2 \right)$$

$$= \frac{\pi^2}{3} - \frac{5}{2}$$

□

Q3 $f(x) = e^x$ on $[-\pi, \pi]$.

(a) $$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-in)x}\,dx$$

$$= \frac{1}{2\pi(1-in)} e^{(1-in)x}\Big|_{x=-\pi}^{\pi}$$

$$= \frac{(-1)^n}{2\pi(1-in)}\left(e^{\pi} - e^{-\pi}\right).$$

(b) Compute $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{1+n^2}$.

$$e^{\pi} = f(\pi) \overset{\times}{=} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{2\pi(1-in)} e^{in\pi} = \sum_{n=-\infty}^{\infty} \frac{e^{\pi} - e^{-\pi}}{2\pi(1+n^2)}$$

Approach 1:

$\hat{f}(n) = O\left(\frac{1}{|n|}\right)$ and f has a jump discontinuity at π

$$S_N f(\pi) \to \frac{1}{2}\left(f(\pi^+) + f(\pi^-)\right) = \frac{e^{\pi} + e^{-\pi}}{2}.$$

$$\frac{e^{\pi}+e^{-\pi}}{2} = \sum_{n=-\infty}^{\infty} \frac{e^{\pi}-e^{-\pi}}{2\pi(1+n^2)}$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} = \pi \frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}}.$$

Approach 2.

By Parseval Identity,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$$

$$\frac{1}{4\pi}(e^{2\pi}-e^{-2\pi}) \qquad \qquad \frac{(e^{\pi}-e^{-\pi})^2}{4\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{1+n^2}$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} = \pi \cdot \frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}}.$$

$\square$

Q5 whether $\exists$ Riemann integrable function f on $[-\pi, \pi]$ s.t. $\hat{f}(n) = \begin{cases} \frac{1}{\log n}, & n \geq 2, \\ 0, & \text{otherwise}. \end{cases}$

Approach 1

$$f * P_r(0) = A_r(f)(0) \leq \sup |f|$$

$$f * F_N(0) = \sigma_N(f)(0) \leq \sup |f|$$

$$\sum_{n=1}^{\infty} \frac{1}{n}, \quad \sum_{n=2}^{\infty} \frac{1}{\log n} = \infty.$$

Contradiction!

Approach 2 Suppose $f \sim \sum_{n=2}^{\infty} \frac{1}{\log n} e^{inx}$

By Parseval Identity,

$$\infty = \sum_{n=2}^{\infty} \frac{1}{\log^2 n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2 dx < \infty$$

$$\sum_{n=2}^{\infty} \frac{1}{\log^2 n} \geq \sum_{n=2}^{\infty} \frac{1}{n \log n}$$

$$\int_1^N \frac{1}{x \log x} dx = \int_1^{\log N} \frac{1}{y} dy \qquad y = \log x$$

$= \log\log N \to \infty$ as $N \to \infty$.

$\square$

Q6 $$\hat{f}(n) = \begin{cases} \frac{1}{2}, & n = 0 \\ \frac{1}{(2k+1)\pi i}, & n = 2k+1 \\ 0, & \text{otherwise}. \end{cases}$$

$$\hat{g}(n) = c_n$$

Find $h \sim \sum_{k=-\infty}^{\infty} \frac{c_{2k+1}}{2k+1} e^{i(2k+1)x}$

$$\widehat{f * g}(n) = \hat{f}(n) \cdot \hat{g}(n).$$

$$\pi i \widehat{f*g}(n) = \pi i \hat{f}(n)\hat{g}(n) = \begin{cases} \frac{c_{2k+1}}{(2k+1)\pi \cancel{i}}, & n = 2k+1. \\ \frac{1}{2} c_0 \pi i, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$h(x) = \pi i f * g(x) - \frac{1}{2} c_0 \pi i.$$

$\square$

Q7 If f, g are 2π-periodic Riemann integrable functions,

then $\lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} f(x)g(nx)\,dx = \hat{f}(0)\hat{g}(0)$.

$\forall\, \varepsilon>0$,

Pf: $\left|\frac{1}{2\pi}\int_0^{2\pi} f(x)g(nx)\,dx - \hat{f}(0)\hat{g}(0)\right|$

$= \left|\frac{1}{2\pi}\int_0^{2\pi} f(x)\big(g(nx) - S_N g(nx)\big)\,dx\right| +$

(I)

$\left|\frac{1}{2\pi}\int_0^{2\pi} f(x)\, S_N g(nx)\,dx - \hat{f}(0)\hat{g}(0)\right|$

(II)

$(\mathrm{I}) = \left|\frac{1}{2\pi}\int_0^{2\pi} f(x)\big(g(nx) - S_N g(nx)\big)\,dx\right|$

by Cauchy-Schwarz

$\leq \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\,dx\right)^{\frac{1}{2}} \left(\frac{1}{2\pi}\int_0^{2\pi} |g(nx) - S_N g(nx)|^2\,dx\right)^{\frac{1}{2}}$

$= \|f\|_2\, \|g - S_N g\|_2$

$$\frac{1}{2\pi}\int_0^{2\pi}|g(nx)-S_N g(nx)|^2\,dx$$

$$=\frac{1}{2\pi}\int_0^{2n\pi}|g(y)-S_N g(y)|^2\,\frac{dy}{n}$$

$y = nx$

$$=\frac{1}{2\pi n}\sum_{k=1}^{n}\int_{2(k-1)\pi}^{2k\pi}|g(y)-S_N g(y)|^2\,dy$$

$$=\frac{1}{2\pi}\int_0^{2\pi}|g(y)-S_N g(y)|^2\,dy=\|g-S_N g\|_2^2$$

$$(\mathrm{II}) = \left|\frac{1}{2\pi}\int_0^{2\pi} f(x)\,S_N g(nx)\,dx-\hat{f}(0)\hat{g}(0)\right|$$

$$=\left|\frac{1}{2\pi}\int_0^{2\pi} f(x)\Big(\sum_{k=-N}^{N}\hat{g}(k)e^{ikn x}\Big)dx-\hat{f}(0)\hat{g}(0)\right|$$

$$=\left|\sum_{k=-N}^{N}\hat{g}(k)\frac{1}{2\pi}\int_0^{2\pi} f(x)e^{iknx}\,dx-\hat{f}(0)\hat{g}(0)\right|$$

$$=\left|\sum_{k=-N}^{N}\hat{g}(k)\hat{f}(-kn)\,dx-\hat{f}(0)\hat{g}(0)\right|$$

$$=\left|\sum_{\substack{k\neq 0\\ |k|\le N}}\hat{f}(-kn)\hat{g}(k)\right|$$

We first choose N large s.t. $(\mathrm{I}) < \varepsilon$

By Riemann Lebesgue Lemma,

$$|\hat{f}(kn)| \to 0 \quad \text{as} \quad n \to \infty.$$

Choose n large s.t.

$$|\hat{f}(-kn)| < \frac{\varepsilon}{2N}, \quad \forall \ |k| < N.$$

And $|\hat{g}(k)| < C$

$$(\mathrm{II}) < C\varepsilon.$$

$\square$