香港中文大學
The Chinese University of Hong Kong

二零二一至二二年度下學期科目考試
Course Examination 2nd Term, 2021-22

科目編號及名稱
Course Code & Title : **MATH3093 Fourier Analysis**

時間
Time allowed : **3** 小時 hours **00** 分鐘 minutes

Date & Time of Examination: April 27, 2022 @ 13:30pm – 16:30pm

Instructions

- This is an open-book examination. You may **ONLY** refer to printed/written materials during the examination. Assessing information on the internet is not allowed.
- You are allowed to use a calculator in the approved list during the examination.
- You shall take the examination in isolation and shall not communicate with any person during the examination other than the course teacher(s) concerned.

  Please kindly be reminded of the following regulations enforced by the university:

  Honesty in Academic Work: *The Chinese University of Hong Kong places very high importance on honesty in academic work submitted by students, and adopts a policy of zero tolerance on cheating and plagiarism. Any related offence will lead to disciplinary action including termination of studies at the University.*
- There are a total of 100 points.
- Answer **ALL** questions.
- Show all steps clearly in your working. **NO** point will be given if sufficient justification is not provided.
- Only handwritten answers on papers or electronic devices will be accepted. Typed answer will **NOT** be accepted.

- Please follow the instruction of submission below:

  1. Write your answers on papers or electronic devices. Only handwritten answers will be accepted and typed answer will **NOT** be accepted.
  2. Scan or take photos of your work (if you write on papers).
  3. Combine your work into a single pdf file.
  4. Name the pdf file by your student id (e.g. 1155123456.pdf).
  5. You must upload the pdf file to Blackboard before 16:30pm, 27 April, 2022. Mark deduction will be made for late submission.
  6. Please check your file carefully to make sure no missing pages.

Answer **ALL** questions.

1 (15pts) Let $h(x)$ be a 2π-periodic function such that $h(x) = |x|$ on $[-\pi, \pi]$.

(a) Determine the Fourier series of h on $[-\pi, \pi]$.

(b) Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$.

(c) Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4}$.

2 (10pts) Assume that f is 2π-periodic, of class C^1, and satisfies $\int_0^{2\pi} f(t)\, dt = 0$. Show that

$$\int_0^{2\pi} |f(t)|^2\, dt \leq \int_0^{2\pi} |f'(t)|^2\, dt$$

with equality if and only if $f(t) = A\sin t + B\cos t$, where A, B are two constants.

3 (10pts) Let f be a continuous function on $[-\pi, \pi]$. Show that the series $\displaystyle\sum_{0\neq n\in\mathbb{Z}} \frac{\widehat{f}(n)}{n}$ converges, where $\widehat{f}(n)$ is the n-th Fourier coefficient of f.

4 (10 pts) Let f be a function on $\mathbb{R}$ defined by $f(x) = e^{-2\pi|x|}$.

(a) Compute the Fourier transform $\widehat{f}(\xi)$ of f.

(b) Compute the Fourier transform $\widehat{g}(\xi)$ of the function $g(x) = (x^2 + x)f(x)$.

5 (10pts) Let $f \in \mathcal{M}(\mathbb{R})$. Let $(K_\delta)_{\delta>0}$ be a family of functions on $\mathbb{R}$ defined by

$$K_\delta(x) = \frac{1}{\sqrt{\delta}} e^{-\pi x^2/\delta}, \quad x \in \mathbb{R},\ \delta > 0.$$

(a) Prove that $(K_\delta)_{\delta>0}$ is a family of good kernels, as $\delta \to 0$.

(b) Prove that $f * K_\delta(x)$ converges to $f(x)$ uniformly on $\mathbb{R}$, as $\delta \to 0$.

6 (15pts) Use the method of Fourier transform to find the solution $u = u(x,t)$ of the following problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} + cu = 0, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = g(x), & x \in \mathbb{R}, \end{cases}$$

where c is a constant and $g \in \mathcal{S}(\mathbb{R})$.

7 (10pts)

Use the method of Fourier analysis to find a function $f \in \mathcal{M}(\mathbb{R})$ such that

$$\int_{-\infty}^{+\infty} f(x-y)e^{-|y|}\,dy = 2e^{-|x|} - e^{-2|x|}.$$

8 (10pts) The Kolmogrov's distribution function Φ equals

$$\Phi(t) = \sum_{n=-\infty}^{\infty} (-1)^n e^{-2n^2t^2} \qquad \text{(where } t > 0\text{)}.$$

Show that $\displaystyle\lim_{t\to 0}\left(\Phi(t) - \frac{\sqrt{2\pi}}{t}e^{-\frac{\pi^2}{8t^2}}\right) = 0$. (Hint: apply the Poisson summation formula).

9 (10pts) Use the method of Fourier analysis to calculate the following integral.

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)(4+x^2)}\,dx.$$

~~ **End of Paper** ~~