THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050C Mathematical Analysis I

Tutorial 8 (March 13)

The following were discussed in the tutorial this week:

1 Limits of Functions

Definition 1.1. *Let $A \subseteq \mathbb{R}$, and let c be a cluster point of A. For a function $f : A \to \mathbb{R}$, a real number L is said to be a **limit of f at c** if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.*

Example 1.1. *Use the definition of limit to show that $\lim_{x \to 2}(x^2 + 4x) = 12$.*

Example 1.2. *Use the definition of limit to show that $\lim_{x \to 2} \frac{x+5}{x^2-3} = 7$.*

1.1 Theorem (Sequential Criterion)**.** *Let $f : A \to \mathbb{R}$ and let c be a cluster point of A. Then the following are equivalent.*

(i) $\lim_{x \to c} f = L$.

(ii) For every sequence (x_n) in A that converges to c such that $x_n \neq c$ for all $n \in \mathbb{N}$, the sequence $(f(x_n))$ converges to L.

1.2 Divergence Criteria. *Let $f : A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of A.*

*(a) If $L \in \mathbb{R}$, then f does **not** have limit L at c if and only if there exists a sequence (x_n) in A with $x_n \neq c$ for all $n \in \mathbb{N}$ such that the sequence (x_n) converges to c but the sequence $(f(x_n))$ does **not** converge to L.*

*(b) The function f does **not** have a limit at c if and only if there exists a sequence (x_n) in A with $x_n \neq c$ for all $n \in \mathbb{N}$ such that the sequence (x_n) converges to c but the sequence $(f(x_n))$ does **not** converge in $\mathbb{R}$.*

Example 1.3. *Show that the limit $\lim_{x \to 0} \frac{1}{\sqrt{x}}$ $(x > 0)$ does not exist.*

2 Classwork

1. Use the definition of limit to show that $\lim_{x \to -1} \frac{x^2 + 2x + 4}{x + 2} = 3$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

   Show that $\lim_{x \to c} f$ does not exist for every $c \in \mathbb{R}$.