THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050C Mathematical Analysis I

Tutorial 6 (February 27)

The following were discussed in the tutorial this week:

1 Subsequences and the Bolzano-Weierstrass Theorem

Definition 1.1. *Let (x_n) be a sequence of real numbers and let $n_1 < n_2 < \cdots < n_k < \cdots$ be a **strictly increasing** sequence of natural numbers. Then the sequence (x_{n_k}) is called a subsequence of (x_n).*

Example 1.1. *Let (x_n) and (y_n) be given sequences. Let (z_n) be the "shuffled" sequence defined by $z_{2n-1} := x_n, z_{2n} := y_n$. Show that (z_n) is convergent if and only if both (x_n) and (y_n) converge to the same limit.*

Example 1.2. *Suppose that $x_n \geq 0$ for all $n \in \mathbb{N}$ and that $\lim((-1)^n x_n)$ exists. Show that (x_n) converges.*

1.1 Theorem. *Let (x_n) be a sequence of real numbers. Then the following are equivalent:*

(i) *(x_n) does not converge to $x \in \mathbb{R}$.*

(ii) *There exists $\varepsilon_0 > 0$ such that for any $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $n_k \geq k$ and $|x_{n_k} - x| \geq \varepsilon_0$.*

(iii) *There exists $\varepsilon_0 > 0$ and a subsequence (x_{n_k}) of (x_n) such that $|x_{n_k} - x| \geq \varepsilon_0$ for all $k \in \mathbb{N}$.*

1.2 The Bolzano-Weierstrass Theorem. *A bounded sequence of real numbers has a convergent subsequence*

Example 1.3. *Prove that a bounded divergent sequence has two subsequences converging to different limits.*

2 Classwork

1. Show that if (x_n) is unbounded, then there exists a subsequence (x_{n_k}) such that $\lim(1/x_{n_k}) = 0$. (**Hint:** If x_{n_k} has been chosen, find $n_{k+1} \in \mathbb{N}$ such that $n_{k+1} > n_k$ and $|x_{n_{k+1}}| > k + 1$.)
2. Given $e_n := (1 + 1/n)^n$ and $\lim(e_n) = e$. Establish the convergence and find the limit of the following sequences:
   (a) $\left((1 + 1/n^2)^{2n^2}\right)$
   (b) $((1 + 2/n)^n))$ (**Hint:** Note that $(1 + \frac{2}{2n})^{2n}(1 + \frac{2}{2n})^{-1} \leq (1 + \frac{2}{2n-1})^{2n-1}$.)