THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050C Mathematical Analysis I

Tutorial 3 (January 30)

The following were discussed in the tutorial this week:

1 Applications of the Supremum Property

Definition 1.1. *Let S be a nonempty subset of $\mathbb{R}$ that is bounded above. Then $u \in \mathbb{R}$ is said to be a* ***supremum*** *of S if*

(i) $s \le u$ for all $s \in S$;

(ii) for any $\varepsilon > 0$, there exists $s_\varepsilon \in S$ such that $u - \varepsilon < s_\varepsilon$.

Example 1.1. *Let S be a nonempty subset of $\mathbb{R}$ that is bounded above, and let a be any number in $\mathbb{R}$. Define the set $a + S := \{a + s : s \in S\}$. Show that*

$$\sup(a + S) = a + \sup S.$$

Example 1.2. *Suppose that A and B are nonempty subsets of $\mathbb{R}$ that satisfy the property:*

$$a \le b \qquad \text{for all } a \in A \text{ and } b \in B.$$

Show that $\sup A \le \inf B$.

Example 1.3. *Let A and B be bounded nonempty subsets of $\mathbb{R}$, and let $A + B := \{a + b : a \in A, b \in B\}$. Prove that*

$$\sup(A + B) = \sup A + \sup B \quad \text{and} \quad \inf(A + B) = \inf A + \inf B.$$

1.1 Archimedean Property. *If $x \in \mathbb{R}$, then there exists $n_x \in \mathbb{N}$ such that $x \le n_x$.*

Example 1.4. *Let $S := \{n/(n + 1) : n \in \mathbb{N}\}$. Find* $\sup S$*, if it exists. Justify your answer.*

2 Classwork

Let $S := \left\{\sqrt{n+1} - \sqrt{n} : n \in \mathbb{N} \cup \{0\}\right\}$. Find $\sup S$ and $\inf S$, if they exist. Justify your answer using ε-argument.

Solution. We will show that $\sup S = 1$ and $\inf S = 0$.

For all $n \in \mathbb{N} \cup \{0\}$, $\sqrt{n+1} - \sqrt{n} = \dfrac{1}{\sqrt{n+1} + \sqrt{n}} \leq 1$. So 1 is an upper bound of S.

Let $\varepsilon > 0$ be given. By the Archimedean Property, there exists $n_\varepsilon \in \mathbb{N}$ such that $n_\varepsilon \geq 1/\varepsilon^2$. Then

$$\sqrt{n_\varepsilon + 1} - \sqrt{n_\varepsilon} = \frac{1}{\sqrt{n_\varepsilon + 1} + \sqrt{n_\varepsilon}} < \frac{1}{\sqrt{n_\varepsilon}} \leq \varepsilon.$$

Hence $\sup S = 1$.

For all $n \in \mathbb{N} \cup \{0\}$, $\sqrt{n+1} - \sqrt{n} \geq 0$. So 0 is a lower bound of S.

Let $\varepsilon > 0$ be given. Then

$$\sqrt{0+1} - \sqrt{0} = 1 > 1 - \varepsilon.$$

Hence $\inf S = 0$. ◀