THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050C Mathematical Analysis I

Tutorial 1 (January 18)

The following were discussed in the tutorial this week:

1 Negation

Example 1.1. *Negate the following statements.*

(a) n is a prime number between 1 and 10.

(b) If n^2 is divisible by 4, then n is divisible by 2.

(c) For any real number x, $x^2 \geq 0$.

(d) For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $1/N < \varepsilon$.

2 Functions

Definition 2.1. *Let A and B be sets. Then a function from A to B is a set f of ordered pairs in $A \times B$ such that for each $a \in A$ there exists a unique $b \in B$ with $(a, b) \in f$. A and B are called the domain and codomain of f, respectively. Denote the function f as $f : A \to B$ and write $b = f(a)$ if and only if $(a, b) \in f$.*

Definition 2.2. *Let $f : A \to B$ be a function.*

(a) Let $E \subseteq A$. Define the direct image of E under f, $f(E)$, by

$$f(E) := \{f(x) : x \in E\}.$$

(b) Let $H \subseteq B$. Define the inverse image of H under f, $f^{-1}(H)$, by

$$f^{-1}(H) := \{x \in A : f(x) \in H\}.$$

Example 2.1. *Let $f : A \to B$ be a function.*

(a) Show that if $E \subseteq A$, then $f^{-1}(f(E)) \supseteq E$. Give an example to show that equality need not hold.

(b) Show that if $H \subseteq B$, then $f\left(f^{-1}(H)\right) \subseteq H$. Give an example to show that equality need not hold.

Definition 2.3. *Let $f : A \to B$ be a function from A to B.*

(a) f is said to be injective if

$$\text{for all } x_1, x_2 \in A, \quad f(x_1) = f(x_2) \implies x_1 = x_2.$$

(b) f is said to be surjective if

$$\text{for all } y \in B, \text{ there exists } x \in A \text{ such that } f(x) = y.$$

(c) f is said to be bijective if f is both injective and surjective. In this case f has an inverse function $f^{-1} : B \to A$ which satisfies

$$b = f(a) \quad \text{if and only if} \quad a = f^{-1}(b).$$

Example 2.2. *Show that the function f defined by $f(x) := x/\sqrt{x^2+1}$, $x \in \mathbb{R}$, is a bijection of $\mathbb{R}$ onto $\{y : -1 < y < 1\}$.*

3 Mathematical Induction

3.1 Well-Ordering Property of $\mathbb{N}$. *Every nonempty subset of $\mathbb{N}$ has a least element.*

3.2 Principle of Mathematical Induction. *Let S be a subset of $\mathbb{N}$ that possesses the following two properties:*

(1) $1 \in S$.

(2) For every $k \in \mathbb{N}$, if $k \in S$, then $k+1 \in S$.

Then we have $S = \mathbb{N}$.

Example 3.1. *Let $\{x_n\}$ be a sequence defined by $x_1 = 1$ and $x_{n+1} = \dfrac{1}{5}(3x_n + 4)$ for all $n \in \mathbb{N}$. Show that $x_n \le x_{n+1} \le 2$ for all $n \in \mathbb{N}$.*

3.3 Principle of Strong Induction. *Let S be a subset of $\mathbb{N}$ that possesses the following two properties:*

(1) $1 \in S$.

(2) For every $k \in \mathbb{N}$, if $\{1, 2, \ldots, k\} \subseteq S$, then $k+1 \in S$.

Then we have $S = \mathbb{N}$.

Remark. (3.1), (3.2) *and* (3.3) *are all equivalent.*

4 Finite and Infinite Sets

Definition 4.1. *(a) The empty set $\emptyset$ is said to have 0 elements.*

(b) If $n \in \mathbb{N}$, a set S is said to have n elements if there exists a bijection from the set $\mathbb{N}_n := \{1, 2, \ldots, n\}$ onto S.

(c) A set is said to be finite if it is either empty or it has n elements for some $n \in \mathbb{N}$.

(d) A set is said to be infinite if it is not finite.

Definition 4.2. *(a) A set S is said to be denumerable (or countably infinite) if there exists a bijection of $\mathbb{N}$ onto S.*

(b) A set S is said to be countable if it is either finite or denumerable.

(c) A set S is said to be uncountable if it is not countable.

Example 4.1. *Show that $\mathbb{Z}$ is denumerable.*

Theorem 4.1. *$\mathbb{N} \times \mathbb{N}$ is denumerable.*

Theorem 4.2. *$\mathbb{Q}$ is denumerable.*

5 Inequalities and Absolute Value

Example 5.1. *Determine the set $A := \{x \in \mathbb{R} : \dfrac{2x+1}{x+2} < 1\}$.*

Definition 5.1. *The absolute value of a real number a, denoted by $|a|$, is defined by*

$$a := \begin{cases} a & \text{if } a > 0, \\ 0 & \text{if } a = 0, \\ -a & \text{if } a < 0. \end{cases}$$

Example 5.2. *Determine the set $B := \{x \in \mathbb{R} : |x| + |x+1| < 2\}$.*

6 Classwork

1. Negate the following statements.
   (a) For any subset S of $\mathbb{N}$, if $S \neq \emptyset$, then there exists $m \in S$ such that $m \leq n$ for all $n \in S$.
   (b) There exists an $\varepsilon > 0$ such that for any natural number N, $|n - m| \geq \varepsilon$ for some $n, m \geq N$.
   (c) For any integer $n \geq 3$, there are no three integers a, b, c that satisfy $a^n + b^n = c^n$.

2. Show that the sum of a rational number and an irrational number is an irrational number.

3. Let $f : A \to B$ be a function.
   (a) Show that if $E \subseteq A$, then $f^{-1}(f(E)) \supseteq E$. Give an example to show that equality need not hold. What if we further assume that f is injective?
   (b) Show that if $H \subseteq B$, then $f\left(f^{-1}(H)\right) \subseteq H$. Give an example to show that equality need not hold. What if we further assume that f is surjective?

4. Let the number x_n be defined as follows: $x_1 := 1$, $x_2 := 2$, and $x_{n+2} := \dfrac{1}{2}(x_n + x_{n+1})$ for all $n \in \mathbb{N}$. Use the Principle of Strong Induction (1.2.5) to show that $1 \leq x_n \leq 2$ for all $n \in \mathbb{N}$.

5. Determine the set $A := \{x \in \mathbb{R} : |x - 3| < |x| + 1\}$.