MATH 2050C Mathematical Analysis I
2018-19 Term 2

Solution to Problem Set 8

4.1-7

Let $f(x) := x^3, x \in \mathbb{R}$. We want to make the difference less than a preassigned $\varepsilon > 0$ by taking x sufficiently close to c. To do so, we note that

$$\left|x^3 - c^3\right| = \left|x^2 + xc + c^2\right| \cdot |x - c|.$$

Moreover, if $|x - c| < 1$, then $|x| < |c| + 1$ so that

$$\left|x^2 + xc + c^2\right| \leq |x|^2 + |cx| + |c|^2 < (|c| + 1)^2 + |c|(|c| + 1) + |c|^2 = 3|c|^2 + 3|c| + 1.$$

Therefore, if$|x - c| < 1$, we have

$$\left|x^3 - c^3\right| < (3|c|^2 + 3|c| + 1) \cdot |x - c|.$$

Consequently, if we choose

$$\delta(\varepsilon) := \min\{1, \frac{\varepsilon}{3|c|^2 + 3|c| + 1}\},$$

then if $0 < |x - c| < \delta(\varepsilon)$, it will follow that

$$\left|x^3 - c^3\right| < (3|c|^2 + 3|c| + 1) \cdot \delta(\varepsilon) < \varepsilon.$$

Since we have a way of choosing $\delta(\varepsilon) > 0$ for an arbitrary choice of $\varepsilon > 0$, we infer that $\lim_{x \to c} f(x) = \lim_{x \to c} x^3 = c^3, \forall c \in \mathbb{R}$.

4.1-10(b)

Let $g(x) := \frac{x+5}{2x+3}, x \in \mathbb{R} \setminus \{-\frac{3}{2}\}$. Then a little algebraic manipulation gives us

$$|g(x) - 4| = \left|\frac{x + 5 - 8x - 12}{2x + 3}\right| = 7\left|\frac{x + 1}{2x + 3}\right|.$$

To get a bound on the coefficient $\frac{7}{|2x+3|}$, we restrict x by the condition $-\frac{5}{4} < x < -\frac{3}{4}$. For x in this interval, we have $\frac{1}{2} < 2x+3 < \frac{3}{2}$ and $\frac{1}{|2x+3|} < 2$ so that

$$|g(x) - 4| < 14\,|x+1|\,.$$

Now for given $\varepsilon > 0$, we choose

$$\delta(\varepsilon) = \min\{\frac{1}{4}, \frac{\varepsilon}{14}\}.$$

Then if $0 < |x+1| < \delta(\varepsilon)$, we have $|g(x) - 4| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, the assertion is proved.

4.1-12(c)

Denote $f(x) := x + \operatorname{sgn}(x), x \in \mathbb{R}$, $a_n = \frac{1}{n}, n \in \mathbb{N}$ and $b_n = -\frac{1}{n}, n \in \mathbb{N}$. Note that $a_n, b_n \neq 0, \forall n \in \mathbb{N}$ and that (a_n) and (b_n) are convergent sequences with common limit 0. Suppose that $\lim_{x\to 0} f(x) = L$ exist, which implies that $L = \lim(f(a_n)) = \lim(f(b_n))$ by Theorem 4.1.8(b) and Theorem 4.1.5. But $f(a_n) = 1 + \frac{1}{n}$ and $f(b_n) = -1 - \frac{1}{n}$ for any $n \in \mathbb{N}$. Thus $\lim f(a_n) = 1$ while $\lim f(b_n) = -1$, a contradiction.

4.1-12(d)

Denote $g(x) := \sin\frac{1}{x^2}, x \in \mathbb{R}\backslash\{0\}$, $a_n = \frac{1}{\sqrt{2n\pi}}, n \in \mathbb{N}$ and $b_n = \frac{1}{\sqrt{(2n+\frac{1}{2})\pi}}, n \in \mathbb{N}$. Note that $a_n, b_n \neq 0, \forall n \in \mathbb{N}$ and that (a_n) and (b_n) are convergent sequences with common limit 0. Suppose that $\lim_{x\to 0} g(x) = L$ exist, which implies that $L = \lim(g(a_n)) = \lim(g(b_n))$ by Theorem 4.1.8(b) and Theorem 4.1.5. But $g(a_n) = \sin 2n\pi = 0$ and $g(b_n) = \sin(2n+\frac{1}{2})\pi = 1$ for any $n \in \mathbb{N}$. Thus $\lim g(a_n) = 0$ while $\lim g(b_n) = 1$, a contradiction.

4.1-15

(a) Given $\varepsilon > 0$, set $\delta(\varepsilon) = \varepsilon$. If $|x| < \delta$, then either $|f(x) - 0| = |x| < \varepsilon$ if x is rational or $|f(x) - 0| = 0 < \varepsilon$ if x is irrational. Thus f has limit $L = 0$ at $x = 0$.

(b) In order to show the divergence, we show that for any $c \neq 0$ there exist two sequences (a_n) and (b_n) converging to c while $\lim f(a_n) \neq \lim f(b_n)$. Denote $I_n = (c, \frac{1}{n} + c), \forall n \in \mathbb{N}$. By The Density Theorem 2.4.8, for each $n \in \mathbb{N}$, there exists RATIONAL number $a_n \in I_n$. Since $c < a_n < \frac{1}{n} + c$, $\lim a_n = c$ by The Squeeze Theorem. Note that $f(a_n) = a_n$ and $\lim f(a_n) = c$. On the other hand, by The Corollary (of Density Theorem) 2.4.9, for each $n \in \mathbb{N}$, there exists IRRATIONAL number $b_n \in I_n$. Similarly we have $\lim b_n = c$. Note that $f(b_n) = 0$ and $\lim f(b_n) = 0$. Since $c \neq 0$, $\lim f(a_n) \neq \lim f(b_n)$, a contradiction.