MATH 2050C Mathematical Analysis I
2018-19 Term 2

Solution to Problem Set 3

2.4-4(a)

For the infimum part, we show that $a \inf S$ is a lower bound of aS and $u \leq a \inf S$ for any lower bound u of aS. Since $\inf S \leq s, \forall s \in S$ and $a > 0$, $a \inf S \leq as, \forall as \in aS$. Thus $a \inf S$ is a lower bound of aS. Suppose u is a lower bound of aS, i.e. $u \leq as, \forall as \in aS$. Thus $u/a \leq s, \forall s \in S$. u/a is a lower bound of S and $u/a \leq \inf S$. We have $u \leq a \inf S$. As u is an arbitrary lower bound, it follows that $a \inf S = \inf(aS)$ by the definition.
For the supremum part, the same idea as above.

2.4-7

To show $\sup(A+B) = \sup A + \sup B$, take any element $a + b \in A + B$. Since $a \leq \sup A$ and $b \leq \sup B$, $a+b \leq \sup A + \sup B$. $\sup A + \sup B$ is an upper bound. From Lemma 2.3.4, for any positive ε, there exist $a_\varepsilon \in A, a_\varepsilon + \varepsilon/2 \geq \sup A$ and $b_\varepsilon \in B, b_\varepsilon + \varepsilon/2 \geq \sup B$. Thus $a_\varepsilon + b_\varepsilon + \varepsilon \geq \sup A + \sup B$. $\sup(A+B) = \sup A + \sup B$ by Lemma 2.3.4 again.
For $\inf(A+B) = \inf A + \inf B$, we apply similar arguments.

2.4-13

- If $x \in \mathbb{Z}$, set $n_x = x + 1$.
- If $x \notin \mathbb{Z}$ and $x > 0$, there exists such n_x by Corollary 2.4.6.
- If $x \notin \mathbb{Z}$ and $x < 0$, consider $-x$ and apply Corollary 2.4.6 again.

To show the uniqueness of n_x. Suppose there are two distinct integers n_x and m_x satisfying $n_x > m_x$ and $n_x - 1 \leq x < n_x$ and $m_x - 1 \leq x < m_x$. Thus we have $n_x - 1 \leq x$ and $-m_x < -x$. Adding these two inequality, $n_x - 1 - m_x < 0$, i.e. $n_x - m_x < 1$. Recalling $n_x > m_x$, we have $0 < n_x - m_x < 1$, while $n_x - m_x$ is an integer. Contradiction.