MATH 2050C Mathematical Analysis I
2018-19 Term 2

Solution to Problem Set 11

5.3-1

(**M1**)By the Maximum-Minimium Theorem 5.3.4, we have $\inf f(I) = f(x_0) > 0$ for some $x_0 \in I$. Denote $\alpha := \inf f(I)$. $f(x) \geq \inf f(I) = \alpha, \forall x \in I$.
(**M2**)The negation is that for any positive number β, there exist $x_\beta \in I$ so that $f(x_\beta) < \beta$. On the contrary, suppose the negation is true and that there exist $x_n \in [a, b]$ satisfying $0 < f(x_n) < 1/n$ for any $n \in \mathbb{N}$. Since $(x_n) \subseteq [a, b]$, there exists a subsequence (x_{n_k}) converges to some $x_0 \in \mathbb{R}$ by Theorem 3.4.8. We further have $x_0 \in [a, b]$ by the closedness of $[a, b]$. As f is continuous and $f(x_{n_k}) < 1/n_k, \forall k \in \mathbb{N}$,

$$f(x_0) = \lim f(x_{n_k}) \leq \lim 1/n_k = 0.$$

This contradicts against the condition $f(x) > 0, \forall x \in [a, b]$.

5.3-4

Given any polynomial $p(x)$ of odd degree, without loss of generality, denote

$$p(x) := \sum\nolimits_{i=0}^{2n+1} a_i x^i, \quad a_{2n+1} = 1, n \in \mathbb{N}.$$

Also denote $M := \max\{|a_0|, \cdots |a_{2n}|\}$. For $x > \max\{2nM + 1, 1\}$, we have

$$\begin{aligned} p(x) &= \sum\nolimits_{i=0}^{2n+1} a_i x^i \\ &\geq x^{2n+1} - M(x^{2n} + \cdots + x + 1) \\ &\geq x^{2n+1} - 2nMx^{2n} \quad (x^k \geq x^l, \forall x \geq 1, k \leq l \in \mathbb{N}) \\ &= x^{2n}(x - 2nM) \\ &> (2nM + 1)^{2n} > 0. \end{aligned}$$

and similarly $p(x) < 0$ for $x < -\max\{2nM + 1, 1\} = \min\{-2nM - 1, -1\}$. Set $R_0 = \max\{2nM + 1, 1\}$ so that $p(R_0)p(-R_0) < 0$. We deduce that there exists at least one real root for $p(x)$ on $[-R_0, R_0]$ by Theorem 5.3.5.

5.3-11

Since $f(a) < 0$, w is well-defined on I. There exists $(x_n) \subseteq W$ so that $\lim(x_n) = w$. As $f(x_n) < 0$ and f is continuous, $f(w) = \lim f(x_n) \leq 0$. Suppose $f(w) < 0$ and denote $\varepsilon_0 = |f(w)|/2$. Note that $w > a$ by the continuity of f and that $f(a) < 0$ and $w < b$ since $f(b) > 0$. There exists $\delta(\varepsilon_0)$ satisfying $(w - \delta(\varepsilon_0), w + \delta(\varepsilon_0)) \subseteq I$ and $|f(x) - f(w)| < \varepsilon_0$, $\forall x \in (w - \delta(\varepsilon_0), w + \delta(\varepsilon_0))$. Thus $f(w + \delta(\varepsilon_0)/2) < f(w) + \varepsilon_0 = f(w)/2 < 0$, contradicting against the definition of w. Hence $f(w) = 0$.

5.4-2

(i) Note that $\forall x, u \in A$, $0 < \frac{1}{x}, \frac{1}{u} \leq 1$. Given $\varepsilon > 0$, set $\delta = \varepsilon/2$. For any $x, u \in A$ satisfying $|x - u| < \delta$,

$$|f(x) - f(u)| = \left|\frac{x^2 - u^2}{x^2 u^2}\right| = \left|\frac{1}{xu}\right| \left|\frac{1}{x} + \frac{1}{u}\right| |x - u| \leq 2|x - u| < 2\delta = \varepsilon.$$

Thus f is uniformly continuous on A.

(ii) Set $\varepsilon_0 = 1$, $(x_n) = (1/n)$ and $(u_n) = (1/(n+1))$. Note that $(x_n), (u_n) \subseteq B$ and $\lim(x_n - u_n) = 0$. Besides,

$$|f(x_n) - f(u_n)| = \left|n^2 - (n+1)^2\right| = 2n + 1 \geq 1.$$

By Criteria 5.4.2(iii), f is not uniformly continuous on B.

5.4-3

(a) Set $\varepsilon_0 = 1$, $(x_n) = (n)$ and $(u_n) = (n + 1/n)$. Note that $(x_n), (u_n) \subseteq A$ and $\lim(x_n - u_n) = 0$. Besides,

$$|f(x_n) - f(u_n)| = \left|n^2 - (n + 1/n)^2\right| = 2 + 1/n^2 \geq 1.$$

By Criteria 5.4.2(iii), f is not uniformly continuous on A.

(b) Set $\varepsilon_0 = 1$, $(x_n) = (1/(2n\pi))$ and $(u_n) = (1/(2n\pi + \pi/2))$. Note that $(x_n), (u_n) \subseteq B$ and $\lim(x_n - u_n) = 0$. Besides,

$$|f(x_n) - f(u_n)| = |0 - 1| = 1 \geq 1.$$

By Criteria 5.4.2(iii), f is not uniformly continuous on B.