MATH 2050C Mathematical Analysis I
2018-19 Term 2

Solution to Problem Set 10

5.1-11

Given $\varepsilon > 0$ and fixed $c \in \mathbb{R}$, set $\delta = \frac{\varepsilon}{K}$. For any x satisfying $|x - c| < \delta$, we have

$$|f(x) - f(c)| \leq K\,|x - c| < K\delta = \varepsilon.$$

Hence f is continuous at $x = c$. Since c is arbitrary, f is continuous on $\mathbb{R}$.

5.1-12

Fix $x \in \mathbb{R}$. By The Density Theorem, there exists (x_n) so that

$$x_n \in \mathbb{Q} \cap (x + \frac{1}{n+1}, x + \frac{1}{n}), \quad \forall n \in \mathbb{N}.$$

Thus $\lim(x_n) = x$. Since f is continuous and $f(x_n) = 0, \forall n \in \mathbb{N}$,

$$f(x) = \lim f(x_n) = 0.$$

Since x is arbitrary, we have $f(x) = 0, \forall x \in \mathbb{R}$.

5.2-4

Given $c \in \mathbb{R} \setminus \mathbb{Z}$, we have $k < c < k + 1$ for some $k \in \mathbb{Z}$ and $[[c]] < c < [[c]] + 1$. Thus $[[c]] = k$ and $[[x]] = k$ for all $x \in (k, k + 1)$. Thus $[[x]]$ is continuous on $\mathbb{R} \setminus \mathbb{Z}$ and $f(x) = x - [[x]]$ is also continuous on $\mathbb{R} \setminus \mathbb{Z}$ by Theorem 5.2.1(a). Given $x \in \mathbb{Z}$, $x = k$ for some $k \in \mathbb{Z}$. Define $(a_n) = (k - \frac{1}{n})$ and $(b_n) = (k + \frac{1}{n})$. We have $f(a_n) = a_n - [[a_n]] = k - \frac{1}{n} - (k - 1) = 1 - \frac{1}{n}$ and $f(b_n) = b_n - [[b_n]] = k + \frac{1}{n} - k = \frac{1}{n}$. Thus $1 = \lim f(a_n) \neq \lim f(b_n) = 0$. $f(x)$ is discontinuous on $\mathbb{Z}$. In conclusion, the points of continuity of $x - [[x]]$ is $\mathbb{R} \setminus \mathbb{Z}$.

5.2-8

Denote $h := f - g$. $h(x)$ is continuous on $\mathbb{R}$ by Theorem 5.2.1(a). $h(r) = f(r) - g(r) = 0, \forall r \in \mathbb{Q}$. Apply the result of Exercise 5.1-12. We have $h(x) = 0, \forall x \in \mathbb{R}$, i.e. $f(x) = g(x), \forall x \in \mathbb{R}$.

5.2-15

Define $l(x) := \min\{f(x), g(x)\}$. We claim that for any $x \in \mathbb{R}$,

$$h(x) + l(x) = f(x) + g(x); \quad h(x) - l(x) = |f(x) - g(x)|.$$

Given $x \in \mathbb{R}$, if $f(x) \leq g(x)$, $h(x) = g(x)$ and $l(x) = f(x)$ and the formulas follow. If $g(x) < f(x)$, similarly we have the same formulas. We finish the claim. Hence

$$h(x) = \frac{1}{2}[h(x) - l(x) + h(x) + l(x)] = \frac{1}{2}[f(x) + g(x) + |f(x) - g(x)|].$$

Provided that both f and g are continuous at $x = c$, h is continuous at $x = c$ by Theorem 5.2.1(a) and Theorem 5.2.4(a).