MATH 2028 Green's Theorem

GOAL: Discuss Green's Theorem in $\mathbb{R}^2$ and its applications

We now consider the special case of line integrals in $\mathbb{R}^2$.

Notation: Given any vector field $F : \mathcal{U} \to \mathbb{R}^2$ defined on an open set $\mathcal{U} \subseteq \mathbb{R}^2$, we write in components:

$$F = (P, Q) \quad \text{and} \quad d\vec{r} = (dx, dy)$$

THEN: $F \cdot d\vec{r} = \underbrace{P\,dx + Q\,dy}_{\text{"1-form" on } \mathcal{U}}$

Hence, we shall also use the notation for line integrals:

$$\int_C F \cdot d\vec{r} = \int_C P\,dx + Q\,dy$$

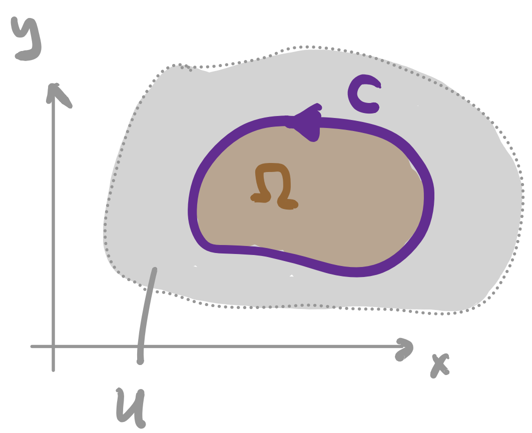
We are now ready to state the "1^{st} Fundamental Theorem for Multi-variable Calculus" for line integrals in $\mathbb{R}^2$.

<u>Green's Theorem</u>: Let $F : \mathcal{U} \to \mathbb{R}^2$ be a C^1 vector field defined on an open set $\mathcal{U} \subseteq \mathbb{R}^2$.

<u>THEN</u>: for any compact domain $\Omega \subseteq \mathcal{U}$ with piecewise C^1 boundary $C = \partial\Omega$, oriented "positively" s.t. the region Ω always lie on the left of C, we have

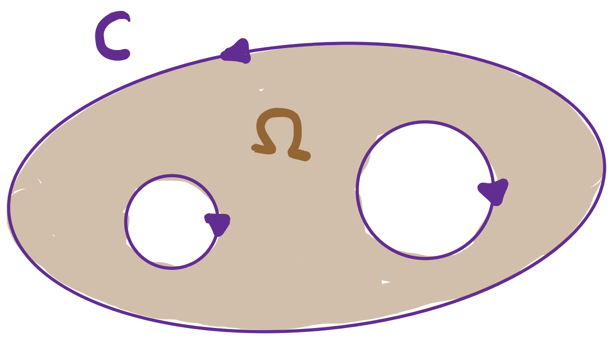
$$\boxed{\int_C P\,dx + Q\,dy = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$

where $F = (P, Q)$ in components.



<u>Remarks</u>: (1) Since F is a C^1 vector field, the components P and Q are C^1 functions, and thus $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is a cts function, hence integrable on $\mathcal{U}$.

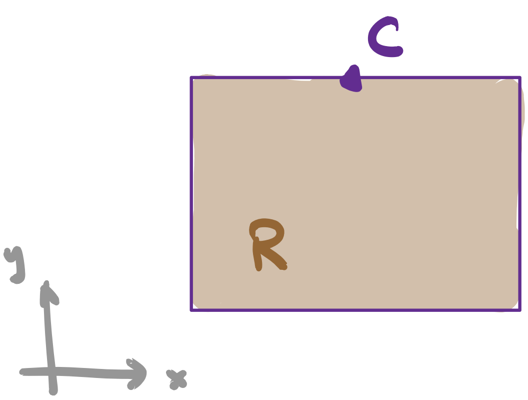
(2) The boundary curve $C = \partial\Omega$ can be "disconnected". For example.



Proof of Green's Theorem for rectangles:

We now give the proof when Ω is a rectangle $R \subseteq \mathbb{R}^2$. The general case will be proved later.

<u>Main Idea</u>: Fubini's Thm!



$$R = [a,b] \times [c,d] \subseteq \mathbb{R}^2$$

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_R \frac{\partial Q}{\partial x}\, dA - \iint_R \frac{\partial P}{\partial y}\, dA$$

Fubini:

$$= \int_c^d \int_a^b \frac{\partial Q}{\partial x}\, dx\, dy - \int_a^b \int_c^d \frac{\partial P}{\partial y}\, dy\, dx$$

Fund. Thm. of Calculus

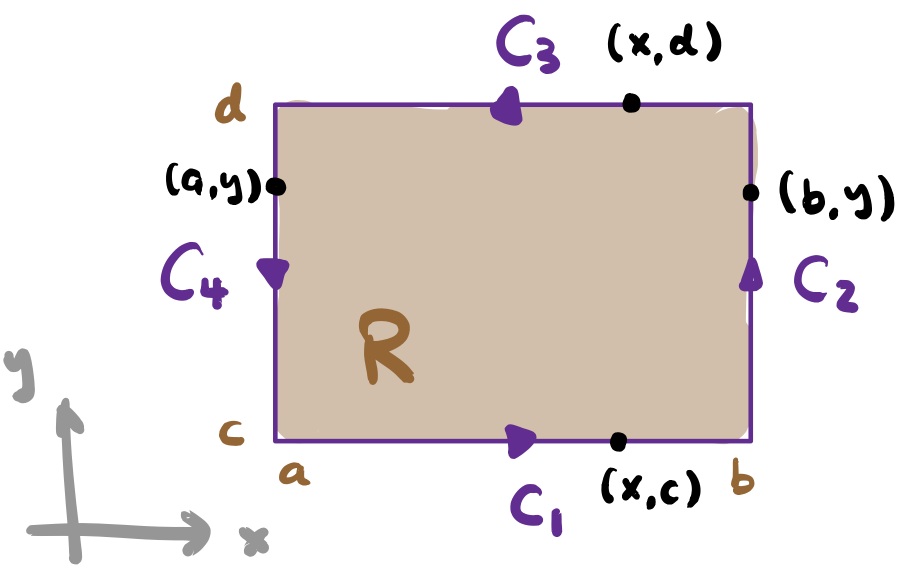
$$= \int_c^d Q(b,y) - Q(a,y)\, dy - \int_a^b P(x,d) - P(x,c)\, dx$$

$$= \underbrace{\int_a^b P(x,c)\, dx}_{= \int_{C_1} F\cdot d\vec{r}} + \underbrace{\int_c^d Q(b,y)\, dy}_{= \int_{C_2} F\cdot d\vec{r}} - \underbrace{\int_a^b P(x,d)\, dx}_{= -\int_{C_3} F\cdot d\vec{r}}$$

$$- \underbrace{\int_c^d Q(a,y)\, dy}_{= -\int_{C_4} F\cdot d\vec{r}}$$

See the picture

$$= \int_C P\, dx + Q\, dy$$



$$C = \partial\Omega = C_1 \cup C_2 \cup C_3 \cup C_4$$

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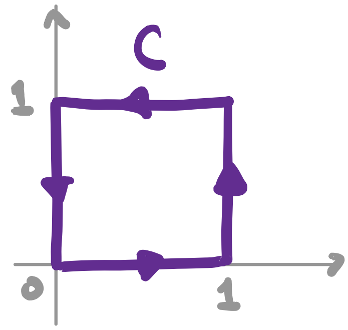
Example 1 : Evaluate $\int_C F \cdot d\vec{r}$ where

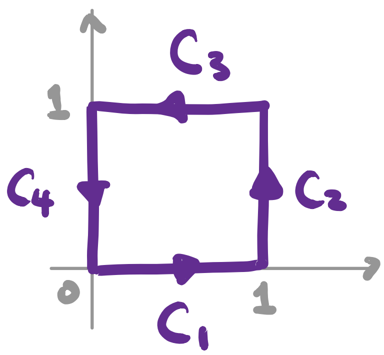
$F = (x^2 - y^2, 2xy)$



and $C \subseteq \mathbb{R}^2$ is the unit square

as shown oriented counterclockwise.

Solution 1 : Direct computation



$C = C_1 \cup C_2 \cup C_3 \cup C_4$

Parametrize C_1 by $\gamma_1(t) = (t, 0),\ 0 \le t \le 1$

$$\int_{C_1} F \cdot d\vec{r} = \int_0^1 t^2\, dt = \frac{1}{3}$$

Parametrize C_2 by $\gamma_2(t) = (1, t),\ 0 \le t \le 1$

$$\int_{C_2} F \cdot d\vec{r} = \int_0^1 2t\, dt = 1$$

Parametrize $-C_3$ by $\gamma_3(t) = (t, 1),\ 0 \le t \le 1$

$$\int_{C_3} F \cdot d\vec{r} = -\int_0^1 (t^2 - 1)\, dt = \frac{2}{3}$$

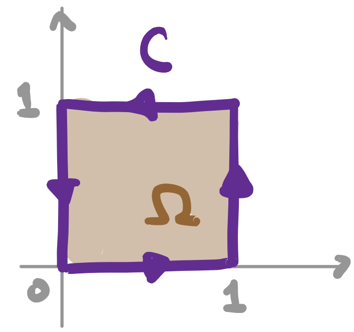
Parametrize $-C_4$ by $\gamma_4(t) = (0, t),\ 0 \le t \le 1$

$$\int_{C_4} F \cdot d\vec{r} = -\int_0^1 0\, dt = 0$$

Thus, $\int_C F \cdot d\vec{r} = \frac{1}{3} + 1 + \frac{2}{3} + 0 = 2$ \#

Solution 2 : Use Green's Theorem

Note that $C = \partial\Omega$ with the "positive orientation as shown :



By Green's Theorem,

$P = x^2 - y^2$
$Q = 2xy$

$$\int_C F \cdot d\vec{r} = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \iint_\Omega [2y - (-2y)] \, dA$$

$$= \int_0^1 \int_0^1 4y \, dy \, dx = 2$$

We can compute the area using Green's Theorem.

Suppose $\Omega \subseteq \mathbb{R}^2$ is a bdd region with piecewise C^1 boundary. THEN:

$$\text{Area}(\Omega) = \int_{\partial\Omega} x \, dy = \int_{\partial\Omega} -y \, dx = \frac{1}{2} \int_{\partial\Omega} -y \, dx + x \, dy$$

Reason: The vector fields

$$F=(0,x)\ ,\quad F=(-y,0)\ ,\quad F=\left(-\frac{y}{2},\frac{x}{2}\right)$$

all satisfy $\dfrac{\partial Q}{\partial x}-\dfrac{\partial P}{\partial y}=1$. Hence, we have

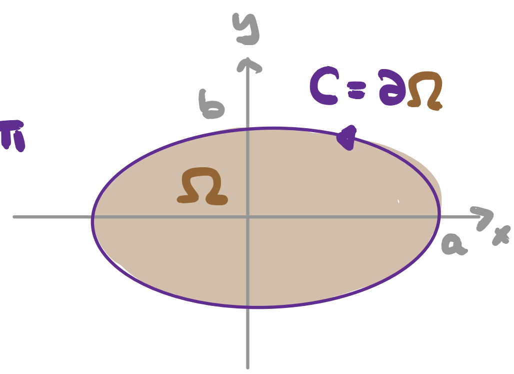
$$\text{Area}(\Omega)=\iint_{\Omega} 1\,dA=\iint_{\Omega}\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\,dA\overset{\text{Green's Thm}}{=}\int_{\partial\Omega}F\cdot d\vec{r}$$

Example 2: Compute the area of the ellipse

$$\Omega:=\left\{(x,y)\ \middle|\ \frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1\right\}\subseteq\mathbb{R}^2$$

Solution: Parametrize C by

$$\gamma(t)=(a\cos t,\ b\sin t),\quad 0\leq t\leq 2\pi$$



$$\text{Area}(\Omega)=\int_C x\,dy$$

$$=\int_0^{2\pi} a\cos t\cdot b\cos t\ dt$$

$$=ab\int_0^{2\pi}\cos^2 t\,dt=\pi ab$$

*

Recall that a vector field $F : \mathcal{U} \to \mathbb{R}^2$ is <u>conservative</u> (on $\mathcal{U}$) if $\exists$ potential function $f : \mathcal{U} \to \mathbb{R}^2$ s.t. $F = \nabla f$.

Write $F = (P, Q)$, then a necessary condition for F to be conservative is the following:

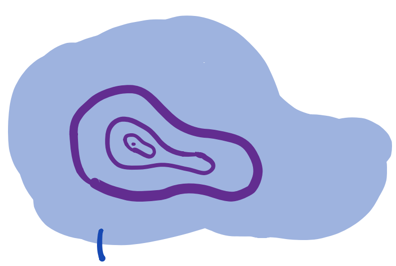
<u>Compatibility Condition:</u> $\boxed{\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}}$ $(*)$

<u>Q</u>: When is $(*)$ sufficient ?

<u>A</u>: when $\mathcal{U}$ is "simply connected".

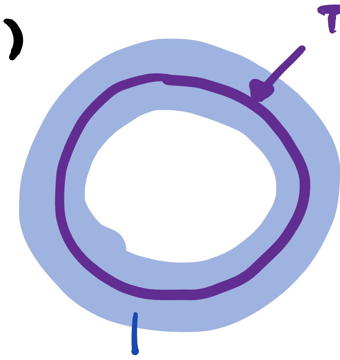
<u>Def$^{\underline{n}}$</u>: A subset $\mathcal{U} \subseteq \mathbb{R}^2$ is simply connected if it is connected and every closed curve in $\mathcal{U}$ can be continuously shrunk to a point without leaving $\mathcal{U}$.

<u>E.g.</u>)



$\mathcal{U}$ simply connected

<u>non. E.g.</u>)



This closed curve CANNOT be shrunk to a point continuously <u>IN</u> $\mathcal{U}$

$\mathcal{U}$ NOT simply connected

Prop: If $F : \mathcal{U} \to \mathbb{R}^2$ is a C^1 vector field defined on an open set $\mathcal{U} \subseteq \mathbb{R}^2$ which is simply connected and that $(*)$ is satisfied,

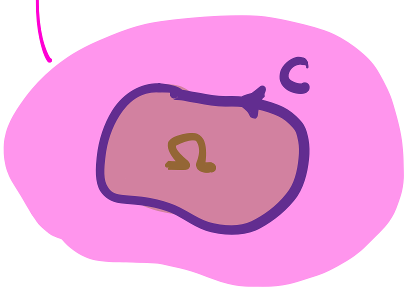
THEN: F is conservative on $\mathcal{U}$.

Proof: By Thm in L12, it suffices to show that

$$\int_C F \cdot d\vec{r} = 0$$ for ALL closed curve $C \subseteq \mathcal{U}$.

Take ANY such closed curve $C \subseteq \mathcal{U}$.

$\mathcal{U}$ simply connected $\Rightarrow$ C can be continuously shrunk to a point in $\mathcal{U}$



$\Rightarrow$ C bounds a region $\Omega \subseteq \mathcal{U}$ s.t. $C = \partial\Omega$

By Green's Theorem, writing $F = (P, Q)$.

$$\int_C F \cdot d\vec{r} = \iint_\Omega \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{\equiv 0 \text{ using } (*)} dA = 0$$

DONE!

Example 3: Consider the following vector field

$$F = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right) = (P, Q)$$

defined on $\mathcal{U} = \mathbb{R}^2 \setminus \{0\} \subseteq \mathbb{R}^2$. Note that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \left(\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} \right) - \left(-\frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2} \right)$$

$$= \frac{2}{x^2+y^2} - \frac{2(x^2+y^2)}{(x^2+y^2)^2} \equiv 0$$

i.e. F satisfies the compatibility condition (*).

However, F is NOT conservative on $\mathcal{U} = \mathbb{R}^2 \setminus \{0\}$.

Reason: Take $C \subseteq \mathcal{U}$ as the unit circle

parametrized by $\gamma(t) = (\cos t, \sin t)$, $0 \leq t \leq 2\pi$

$$\int_C F \cdot d\vec{r} = \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t) \, dt$$

$$= \int_0^{2\pi} 1 \, dt = 2\pi \neq 0$$

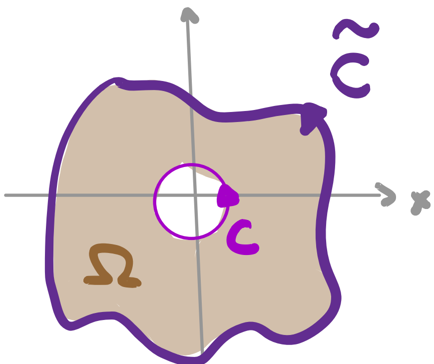
In fact, if we consider any closed curve $\tilde{C} \subseteq \mathcal{U}$

which is "simple", i.e. it does not have any self-intersection, then we still have

$$\int_{\tilde{C}} F \cdot d\vec{r} = 2\pi$$

oriented counterclockwise

Reason:



$\partial\Omega = \tilde{C} \cup -C$

Take any small circle C "inside" the region bounded by $\tilde{C}$ centered at the origin.

THEN: $\tilde{C}$ and C together bound a region $\Omega \subseteq \mathcal{U} = \mathbb{R}^2 \setminus \{0\}$

Since F satisfies $(*)$ everywhere on Ω,

$$0 \overset{(*)}{=} \iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\tilde{C}} F \cdot d\vec{r} - \int_{C} F \cdot d\vec{r}$$

Hence, $\displaystyle\int_{\tilde{C}} F \cdot d\vec{r} = \int_{C} F \cdot d\vec{r} = 2\pi$

direct computation

$\blacksquare$