Solution to Homework 8

Sec. 6.2

17. Note that $\langle T(x), y\rangle = 0$ for any $y \in V$, so we have $T(x) = 0$. But x is arbitrary, so $T = T_0$.

    Suppose $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V and

    $$\langle T(v_i), v_j\rangle = 0$$

    for $1 \le i, j \le n$. Note that x and y can be expressed as linear combinations of v_is. By the linearity of T and the inner product $\langle \cdot, \cdot\rangle$, one can easily show that

    $$\langle T(x), y\rangle = 0$$

    for all $x, y \in V$. Hence, by the above argument, we have $T = T_0$.

18. We show that $W_e^{\perp} \subset W_o$ and $W_e^{\perp} \supset W_o$.

    For any $h \in W_e^{\perp}$, we decompose h into f and g in this way.

    $$f(t) = \frac{1}{2}(h(t) + h(-t))$$

    $$g(t) = \frac{1}{2}(h(t) - h(-t))$$

    Obviously, $h = f + g$ and one can check that f is an even function, while g is an odd function. By assumption, we have $\langle h, f\rangle = 0$ as $f \in W_e$, which means

    $$0 = \langle f + g, f\rangle = \langle f, f\rangle + \langle g, f\rangle = \|f\|^2$$

    because $\langle g, f\rangle = \int_{-1}^{1} f(t)g(t)dt = 0$ as $f(t)g(t)$ is an even function. Hence, we have $h = f + g = g \in W_o$ and $W_e^{\perp} \subset W_o$.

    On the other hand, for any $k \in W_o$, we have

    $$\langle k, f\rangle = \int_{-1}^{1} k(t)f(t)dt = 0$$

    for any $f \in W_e$ as $k(t)f(t)$ is an even function. Hence, we have $k \in W_e^{\perp}$ and $W_e^{\perp} \supset W_o$.

Sec. 6.3

2. (b) Let $\beta = \{v_1, v_2\}$ be the standard basis for $\mathbb{C}^2$. Obviously, β is an orthonormal basis. Then we have

$$y = \sum_{i=1}^{2} \overline{g(v_i)} v_i = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

3. (b) Let $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2$. Consider $\langle z, T^*(x) \rangle$, we have the following.

$$\begin{aligned} \langle z, T^*(x) \rangle &= \langle T(z), x \rangle \\ &= \left\langle \begin{pmatrix} 2z_1 + iz_2 \\ (1-i)z_1 \end{pmatrix}, \begin{pmatrix} 3-i \\ 1+2i \end{pmatrix} \right\rangle \\ &= (5-i)z_1 + (-1+3i)z_2 \end{aligned}$$

Hence, we see that $T^*(x) = \begin{pmatrix} 5+i \\ -1-3i \end{pmatrix}$.

(c) Similarly, let $g(t) = at + b \in P_1(\mathbb{R})$.

$$\begin{aligned} \langle g, T^*(f) \rangle &= \langle T(g), f \rangle \\ &= \langle a + 3(at+b), 4 - 2t \rangle \\ &= \int_{-1}^{1} \big(-6at^2 + (10a - 6b)t + 4(a+3b)\big)\, dt \\ &= 4a + 24b \end{aligned}$$

By letting $T^*(f) = ct + d$.

$$\langle g, T^*(f) \rangle = \int_{-1}^{1} (at+b)(ct+d)dt = \frac{2}{3}ac + 2bd$$

We see that $c = 6$ and $d = 12$. Hence, $T^*(f) = 6t + 12$.

6. Obviously, we have

$$U_1^* = (T + T^*)^* = T^* + (T^*)^* = T + T^* = U_1$$

and

$$U_2^* = (TT^*)^* = (T^*)^* T^* = TT^* = U_2.$$

8. Note that T is invertible, so T^{-1} exists.

$$T^*(T^{-1})^* = (T^{-1}T)^* = I^* = I$$

$$(T^{-1})^* T^* = (TT^{-1})^* = I^* = I$$

Hence, $T*$ is invertible and $(T^*)^{-1} = (T^{-1})^*$

9. Suppose W is finite-dimensional subspace of V and $V = W \oplus W^{\perp}$. For any $x, y \in V$, we have $x = x_1 + x_2$ and $y = y_1 + y_2$, where $x_1, y_1 \in W$ and $x_2, y_2 \in W^{\perp}$. So we have $\langle x_1, y_2 \rangle = 0 = \langle x_2, y_1 \rangle$. We want to show that $T(x) = T^*(x)$ for all $x \in V$.

$$\begin{aligned}
\langle T^*(x), y \rangle &= \langle x, T(y) \rangle \\
&= \langle x_1 + x_2, y_1 \rangle \\
&= \langle x_1, y_1 \rangle
\end{aligned}$$

Similarly, we have the following.

$$\begin{aligned}
\langle T(x), y \rangle &= \langle x_1, y_1 + y_2 \rangle \\
&= \langle x_1, y_1 \rangle \\
&= \langle T^*(x), y \rangle
\end{aligned}$$

Since the above holds for any $y \in V$ and x is arbitrary, we see that $T = T^*$.

10. Note that from Exercise 20 in Sec. 6.1, we have the following.

$$\langle x, y \rangle = \frac{1}{4} \|x + y\|^2 - \frac{1}{4} \|x - y\|^2 \text{ if } \mathbb{F} = \mathbb{R}$$

$$\langle x, y \rangle = \frac{1}{4} \sum_{k=1}^{4} i^k \left\|x + i^k y\right\|^2 \text{ if } \mathbb{F} = \mathbb{C}$$

Now if $\|T(x)\| = \|x\|$ for all $x \in V$. For $\mathbb{F} = \mathbb{R}$, we have the following.

$$\begin{aligned}
\langle T(x), T(y) \rangle &= \frac{1}{4} \|T(x) + T(y)\|^2 - \frac{1}{4} \|T(x) - T(y)\|^2 \\
&= \frac{1}{4} \|T(x + y)\|^2 - \frac{1}{4} \|T(x - y)\|^2 \\
&= \frac{1}{4} \|x + y\|^2 - \frac{1}{4} \|x - y\|^2 \\
&= \langle x, y \rangle
\end{aligned}$$

Similarly, for $\mathbb{F} = \mathbb{C}$, we have the following.

$$\begin{aligned}
\langle T(x), T(y) \rangle &= \frac{1}{4} \sum_{k=1}^{4} i^k \left\|T(x) + i^k T(y)\right\|^2 \\
&= \frac{1}{4} \sum_{k=1}^{4} i^k \left\|T(x + i^k y)\right\|^2 \\
&= \frac{1}{4} \sum_{k=1}^{4} i^k \left\|x + i^k y\right\|^2 \\
&= \langle x, y \rangle
\end{aligned}$$

If $\langle T(x), T(y) \rangle = \langle x, y \rangle$, we simply take $y = x$ and the result follows.

13. (a) Obviously, if $x \in N(T)$, we have

$$T^*T(x) = T^*(0) = 0.$$

So $x \in N(T^*T)$. Conversely, if $x \in N(T^*T)$, we have

$$\|T(x)\|^2 = \langle T(x), T(x)\rangle = \langle x, T^*T(x)\rangle = \langle x, 0\rangle = 0.$$

So $T(x) = 0$ and $x \in N(T)$. Note that T^*T is also a linear operator on V and V is of finite dimension. By the dimension of rank and nullity, we see that $\text{rank}(T^*T) = \text{rank}(T)$.

(b) First, we show that $\text{rank}(A^*) = \text{rank}(A)$. Note that $\text{rank}(A^t) = \text{rank}(A)$ as the dimension of column space equals that of row space. Also, we have $\text{rank}(\overline{A}) = \text{rank}(A)$ as $\{v_1, v_2, \ldots, v_n\}$ are linearly independent if and only if $\{\overline{v_1}, \overline{v_2}, \ldots, \overline{v_n}\}$ are linearly independent.

$$\sum b_i\overline{v_i} = \sum \overline{a_i v_i} = \overline{\sum a_i v_i}, \text{ with } a_i = \overline{b_i}$$

As $A^* = \overline{A^t}$, we have $\text{rank}(A^*) = \text{rank}(A)$. Then we have

$$\text{rank}([T]_\beta^*) = \text{rank}([T]_\beta).$$

But $[T^*]_\beta = [T]_\beta^*$, so we have

$$\text{rank}([T^*]_\beta) = \text{rank}([T]_\beta).$$

In other words, $\text{rank}(T^*) = \text{rank}(T)$.
Using (a), we have $\text{rank}(TT^*) = \text{rank}(T^*)$ by considering T^* instead of T. By the above argument, we have

$$\text{rank}(TT^*) = \text{rank}(T^*) = \text{rank}(T).$$

(c) From (a) and (b), we have the following.

$$\text{rank}(L_A(L_A)^*) = \text{rank}((L_A)^*L_A) = \text{rank}(L_A)$$

Using the fact that $L_{A^*} = (L_A)^*$ and $L_AL_B = L_{AB}$, we have the result.

$$\text{rank}(L_{AA^*}) = \text{rank}(L_{A^*A}) = \text{rank}(L_A)$$

$$\text{rank}(AA^*) = \text{rank}(A^*A) = \text{rank}(A)$$

15. (a) Note for a fixed $y \in W$, we may regard $\langle T(x), y\rangle_2$ as a linear transformation from V to $\mathbb{F}$. Then there is a unique $z \in V$ such that

$$\langle T(x), y\rangle_2 = \langle x, z\rangle_1$$

for all $x \in V$. We may define $T^*(y) = z$. As $z \in V$ exists and is unique for any given $y \in W$, we see that $T^* : W \to V$ is well-defined. Hence, we now have

$$\langle T(x), y\rangle_2 = \langle x, T^*(y)\rangle_1$$

for any $x \in V$ and $y \in W$. If there is a transformation $U : W \to V$ satisfying the same condition, we have

$$\langle x, U(y)\rangle_1 = \langle T(x), y\rangle_2 = \langle x, T^*(y)\rangle_1$$

for any $x \in V$ and $y \in W$, which means $U = T^*$.
To check the linearity of T^*, we have the following.

$$\begin{aligned}\langle x, T^*(y + cz)\rangle_1 &= \langle T(x), y + cz\rangle_2 \\ &= \langle T(x), y\rangle_2 + \bar{c}\,\langle T(x), z\rangle_2 \\ &= \langle x, T^*(y)\rangle_2 + \bar{c}\,\langle x, T^*(z)\rangle_2 \\ &= \langle x, T^*(y) + cT^*(z)\rangle_1\end{aligned}$$

Since x is arbitrary, we have $T^*(y + cz) = T^*(y) + cT^*(z)$.

(b) Let $\beta = \{v_1, v_2, \dots, v_n\}$ and $\gamma = \{w_1, w_2, \dots, w_m\}$ be orthonormal bases for V and W respectively. Consider $T(v_j)$ and $T^*(w_j)$, we have the following.

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i, \quad T^*(w_j) = \sum_{i=1}^{n} b_{ij} v_i$$

Note that $[T]_\beta^\gamma = \left(a_{ij}\right)$ and $[T^*]_\gamma^\beta = \left(b_{ij}\right)$ Now that $\langle x, T^*(y)\rangle_1 = \langle T(x), y\rangle_2$, we have the following.

$$\overline{b_{ji}} = \langle v_j, T^*(w_i)\rangle_1 = \langle T(v_j), w_i\rangle_2 = a_{ij}$$

Hence, we see that $[T^*]_\gamma^\beta = ([T]_\beta^\gamma)^*$

(c) Again we have

$$\text{rank}([T^*]_\gamma^\beta) = \text{rank}(([T]_\beta^\gamma)^*) = \text{rank}([T]_\beta^\gamma).$$

Hence, we have $\text{rank}(T^*) = \text{rank}(T)$.

(d) Using the fact that $\langle a, b\rangle = \overline{\langle b, a\rangle}$ and the property of adjoint.

$$\langle T^*(x), y\rangle_1 = \overline{\langle y, T^*(x)\rangle}_1 = \overline{\langle T(y), x\rangle}_2 = \langle x, T(y)\rangle_2$$

(e) Obviously, if $T(x) = 0$, we have $T^*T(x) = T^*(0) = 0$.
Conversely, if $T^*T(x) = 0$, consider

$$\|T(x)\|^2 = \langle T(x), T(x)\rangle_2 = \langle x, T^*T(x)\rangle_1 = 0$$

and, hence, we have $T(x) = 0$.