

THE CHINESE UNIVERSITY OF HONG KONG  
Department of Mathematics  
**MATH2050C Mathematical Analysis I**  
**Tutorial 8 (March 25)**

**Definition.** Let  $A \subseteq \mathbb{R}$ . A point  $c \in \mathbb{R}$  is said to be a **cluster point** of  $A$  if given any  $\delta > 0$ , there exists  $x \in A$ ,  $x \neq c$  such that  $|x - c| < \delta$ .

*Remarks.* (1) Equivalently,  $c$  is a cluster point of  $A$  if and only if

$$V_\delta(c) \cap A \setminus \{c\} \neq \emptyset \quad \text{for any } \delta > 0.$$

(2) A cluster point of  $A$  may or may not be an element of  $A$ .

**Example 1.** The following are some subsets of  $\mathbb{R}$  and their sets of cluster points.

(a) The set of cluster points of  $\mathbb{N}$  is  $\emptyset$

(b) The set of cluster points of  $(1, 2) \cap \mathbb{Q}$  is  $[1, 2]$ .

**Definition.** Let  $A \subseteq \mathbb{R}$ , and let  $c$  be a cluster point of  $A$ . For a function  $f : A \rightarrow \mathbb{R}$ , a real number  $L$  is said to be a **limit of  $f$  at  $c$**  if, given any  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that if  $x \in A$  and  $0 < |x - c| < \delta$ , then  $|f(x) - L| < \varepsilon$ .

In this case, we write

$$\lim_{x \rightarrow c} f = L, \quad \lim_{x \rightarrow c} f(x) = L \quad \text{or} \quad f(x) \rightarrow L \text{ as } x \rightarrow c.$$

**Theorem.** If  $f : A \rightarrow \mathbb{R}$  and if  $c$  is a cluster point of  $A$ , then  $f$  can have only one limit at  $c$ .

**Example 2.** Use the  $\varepsilon$ - $\delta$  definition of limit to show that  $\lim_{x \rightarrow -3} (x^2 + 4x) = -3$ .

**Solution.** Note that  $|(x^2 + 4x) - (-3)| = |x^2 + 4x + 3| = |x + 1||x + 3|$ .

If  $|x + 3| < 1$ , then  $|x + 1| = |(x + 3) - 2| \leq |x + 3| + 2 < 3$ .

Let  $\varepsilon > 0$  be given. Take  $\delta := \min\{\varepsilon/3, 1\}$ . Now if  $0 < |x - (-3)| < \delta$ , then

$$|(x^2 + 4x) - (-3)| = |x + 1||x + 3| < 3 \cdot \frac{\varepsilon}{3} = \varepsilon.$$

Hence  $\lim_{x \rightarrow -3} (x^2 + 4x) = -3$ . ◀

**Example 3.** Use the  $\varepsilon$ - $\delta$  definition of limit to show that  $\lim_{x \rightarrow 2} \frac{x + 6}{x^2 - 2} = 4$ .

**Solution.** Clearly  $f(x) := \frac{x + 6}{x^2 - 2}$  has a natural domain  $\mathbb{R} \setminus \{\pm\sqrt{2}\}$ , which has 2 as a cluster point.

For  $x \in \mathbb{R} \setminus \{\pm\sqrt{2}\}$ ,

$$|f(x) - 4| = \left| \frac{x+6}{x^2-2} - 4 \right| = \frac{|4x^2 - x - 14|}{|x^2 - 2|} = \frac{|4x+7|}{|x^2-2|} \cdot |x-2|.$$

If  $|x-2| < \frac{1}{2}$ , then

$$\frac{3}{2} < x < \frac{5}{2} \implies \frac{1}{4} < x^2 - 2 < \frac{17}{4},$$

and

$$|4x+7| = |4(x-2) + 15| \leq 4|x-2| + 15 \leq 20.$$

Let  $\varepsilon > 0$  be given. Take  $\delta := \min\left\{\frac{\varepsilon}{80}, \frac{1}{2}\right\}$ . Now if  $0 < |x-2| < \delta$ , then

$$|f(x) - 4| = \frac{|4x+7|}{|x^2-2|} \cdot |x-2| < \frac{20}{1/4} \cdot \frac{\varepsilon}{80} = \varepsilon.$$

Hence  $\lim_{x \rightarrow 2} \frac{x+6}{x^2-2} = 4$ . ◀

## Classwork

Use the  $\varepsilon$ - $\delta$  definition of limit to establish the following limits.

(a)  $\lim_{x \rightarrow 0} \frac{x^2}{|x|} = 0$ .

(b)  $\lim_{x \rightarrow 2} \frac{3x-4}{x^2-3} = 2$ .