THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050C Mathematical Analysis I

Tutorial 5 (March 4)

Theorem 1. *Let $X = (x_n)$, $Y = (y_n)$ and $Z = (z_n)$ be sequences of real numbers that converge to x, y and z, respectively.*

(a) Let $c \in \mathbb{R}$. Then the sequences $X+Y, X-Y, X \cdot Y$, and cX converge to $x+y, x-y, xy$, and cx, respectively.

(b) Suppose further that $z_n \neq 0$ for all $n \in \mathbb{N}$, and $z \neq 0$. Then the sequence X/Z converges to x/z.

Example 1. Apply the above theorem to show the following limits.

(a) $\lim \left(\dfrac{2n+1}{n} \right) = 2.$

(b) $\lim \left(\dfrac{2n+1}{n+5} \right) = 2.$

(c) $\lim \left(\dfrac{2n}{n^2+1} \right) = 0.$

(d) Let $X = (x_n)$ be a sequence of real numbers that converges to $x \in \mathbb{R}$. Let p be a polynomial given by

$$p(t) := a_k t^k + a_{k-1} t^{k-1} + \cdots + a_1 t + a_0,$$

where $k \in \mathbb{N}$ and $a_j \in \mathbb{R}$ for $j = 0, 1, \ldots, k$. Then the sequence $(p(x_n))$ converges to $p(x)$.

(e) Let $X = (x_n)$ be a sequence of real numbers that converges to $x \in \mathbb{R}$. Let r be a rational function (that is, $r(t) := p(t)/q(t)$, where p and q are polynomials). Suppose that $q(x_n) \neq 0$ for all $n \in \mathbb{N}$ and that $q(x) \neq 0$. Then the sequence $(r(x_n))$ converges to $r(x) = p(x)/q(x)$.

Theorem 2. *Let the sequence $X = (x_n)$ converge to x. Then the sequence $(|x_n|)$ of absolute values converges to $|x|$. That is, if $x = \lim(x_n)$, then $|x| = \lim(|x_n|)$.*

Theorem 3. *Let $X = (x_n)$ be a sequence of real numbers that converges to x and suppose that $x_n \geq 0$. Then the sequence $(\sqrt{x_n})$ of positive square roots converges and $\lim(\sqrt{x_n}) = \sqrt{x}$.*

Classwork

1. Show that if X and Y are sequences such that X and $X+Y$ are convergent, then Y is also convergent.

2. If $a > 0$ and $b > 0$, show that $\lim \left(\sqrt{(n+a)(n+b)} - n \right) = (a+b)/2$.

3. Let (x_n) be a sequence of real numbers. Define

$$s_n = \frac{x_1 + x_2 + \cdots + x_n}{n} \qquad \text{for all } n \in \mathbb{N}.$$

   If $\lim(x_n) = 0$, show that $\lim(s_n) = 0$.