THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050C Mathematical Analysis I

Tutorial 3 (February 19)

Definition. Let S be a nonempty subset of $\mathbb{R}$. Suppose S is bounded below. Then $w \in \mathbb{R}$ is said to be an **infimum** of S if it satisfies the conditions:

(i) w is a lower bound of S (that is, $s \geq w$ for all $s \in S$), and

(ii) if v is any lower bound of S, then $w \geq v$.

Here (ii) is equivalent to each of the following conditions:

(ii)' if $v > w$, then there exists $s_v \in S$ such that $s_v < v$.

(ii)" for every $\varepsilon > 0$ there exists $s_\varepsilon \in S$ such that $s_\varepsilon < w + \varepsilon$.

Archimedean Property. *If $x \in \mathbb{R}$, then there exists $n_x \in \mathbb{N}$ such that $x \leq n_x$.*

Example 1. Let $S := \{1/n - 1/m : n, m \in \mathbb{N}\}$. Find $\inf S$ and $\sup S$.

Solution. Since $0 < 1/n \leq 1$ for any $n \in \mathbb{N}$, we have

$$-1 \leq 1/n - 1/m \leq 1 \qquad \text{for any } n, m \in \mathbb{N}.$$

Hence S has a lower bound -1 and an upper bound 1.

First we show that $\inf S = -1$. Suppose $\varepsilon > 0$. By Archimedean Property, there exists $n_\varepsilon \in \mathbb{N}$ such that $n_\varepsilon > 1/\varepsilon$. Now $1/n_\varepsilon - 1/1 \in S$ and

$$1/n_\varepsilon - 1/1 < \varepsilon - 1 = -1 + \varepsilon.$$

Hence $\inf S = -1$.

Next we show that $\sup S = 1$. Let $v < 1$. By Archimedean Property, there exists $m_v \in \mathbb{N}$ such that $m_v > 1/(1-v)$. Now $1/1 - 1/m_v \in S$ and

$$1 - v > 1/m_v \implies 1/1 - 1/m_v > v.$$

Hence $\sup S = 1$. ◀

Example 2. Let S be a nonempty bounded subset of $\mathbb{R}$. For $a \in \mathbb{R}$, let $aS := \{as : s \in S\}$. Prove that if $a > 0$, then

$$\inf(aS) = a \inf S.$$

Solution. For any $s \in S$, we have $s \geq \inf S$, so that $as \geq a \inf S$ since $a > 0$. Hence $a \inf S$ is a lower bound of aS.

Suppose v is any lower bound of aS. Then $v \leq as$ for any $s \in S$. Hence $v/a \leq s$ for any $s \in S$, and thus v/a is a lower bound of S. By the definition of infimum, $v/a \leq \inf S$, so that $v \leq a \inf S$. Therefore $a \inf S$ is the greatest lower bound of aS, that is $\inf(aS) = a \inf S$.

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Classwork

1. Let $S := \left\{\sqrt{n+1} - \sqrt{n} : n \in \mathbb{N} \cup \{0\}\right\}$. Find $\sup S$ and $\inf S$, if they exist. Justify your answer using ε-argument.

2. Let S be a nonempty bounded subset of $\mathbb{R}$. Show that if $b < 0$,

$$\inf(bS) = b \sup S.$$