THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050C Mathematical Analysis I

Tutorial 7 (March 17)

Limit of Functions

Definition. Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is said to be a **cluster point** of A if given any $\delta > 0$, there exists $x \in A$, $x \neq c$ such that $|x - c| < \delta$.

Remarks. (1) Equivalently, c is a cluster point of A if and only if

$$V_\delta(c) \cap A \setminus \{c\} \neq \emptyset \quad \text{for any } \delta > 0.$$

(2) A cluster point of A may or may not be an element of A.

Definition. Let $A \subseteq \mathbb{R}$, and let c be a cluster point of A. For a function $f : A \to \mathbb{R}$, a real number L is said to be a **limit of f at c** if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

In this case, we write

$$\lim_{x\to c} f = L, \quad \lim_{x\to c} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to c.$$

Theorem. *If $f\colon A \to \mathbb{R}$ and if c is a cluster point of A, then f can have only one limit at c.*

Example 1. Use the ε-δ definition of limit to show that $\lim_{x\to -3}(x^2 + 4x) = -3$.

Solution. Observe that $x^2 + 4x$ has a natural domain $\mathbb{R}$, which clearly has -3 as a cluster point.

Note that $|(x^2 + 4x) - (-3)| = |x^2 + 4x + 3| = |x + 1||x + 3|$.

If $|x + 3| < 1$, then $|x + 1| = |(x + 3) - 2| \leq |x + 3| + 2 < 3$.

Let $\varepsilon > 0$ be given. Take $\delta := \min\{\varepsilon/3, 1\}$. Now if $0 < |x - (-3)| < \delta$, then

$$|(x^2 + 4x) - (-3)| = |x + 1||x + 3| < 3 \cdot \frac{\varepsilon}{3} = \varepsilon.$$

Hence $\lim_{x\to -3}(x^2 + 4x) = -3$. ◀

Example 2. Use the ε-δ definition of limit to show that $\lim_{x\to 2} \dfrac{x+6}{x^2-2} = 4$.

Solution. Clearly $f(x) := \dfrac{x+6}{x^2-2}$ has a natural domain $\mathbb{R} \setminus \{\pm\sqrt{2}\}$, which has 2 as a cluster point.

For $x \in \mathbb{R} \setminus \{\pm\sqrt{2}\}$,

$$|f(x) - 4| = \left|\frac{x+6}{x^2-2} - 4\right| = \frac{|4x^2 - x - 14|}{|x^2-2|} = \frac{|4x+7|}{|x^2-2|} \cdot |x-2|.$$

If $|x-2| < \dfrac{1}{2}$, then

$$\frac{3}{2} < x < \frac{5}{2} \implies \frac{1}{4} < x^2 - 2 < \frac{17}{4},$$

and

$$|4x+7| = |4(x-2) + 15| \leq 4|x-2| + 15 \leq 20.$$

Let $\varepsilon > 0$ be given. Take $\delta := \min\left\{\dfrac{\varepsilon}{80}, \dfrac{1}{2}\right\}$. Now if $0 < |x-2| < \delta$, then

$$|f(x) - 4| = \frac{|4x+7|}{|x^2-2|} \cdot |x-2| < \frac{20}{1/4} \cdot \frac{\varepsilon}{80} = \varepsilon.$$

Hence $\lim\limits_{x\to 2} \dfrac{x+6}{x^2-2} = 4$. ◀

Example 3. Use the ε-δ definition of limit to evaluate the limit $\lim\limits_{x\to 4} \dfrac{4-x}{2-\sqrt{|x|}}$.

Solution. For $x \in [0, \infty)\setminus\{4\}$,

$$\frac{4-x}{2-\sqrt{|x|}} = \frac{4-x}{2-\sqrt{x}} = \frac{(2+\sqrt{x})(2-\sqrt{x})}{2+\sqrt{x}} = 2 + \sqrt{x},$$

and thus

$$\left|\frac{4-x}{2-\sqrt{|x|}} - 4\right| = |\sqrt{x} - 2| = \left|\frac{x-4}{\sqrt{x}+2}\right| \leq \frac{|x-4|}{2}.$$

Let $\varepsilon > 0$ be given. Set $\delta = \min\{2\varepsilon, 1\}$. Now, if $0 < |x-4| < \delta$, then $x \in (3,5)\setminus\{4\}$, and hence

$$\left|\frac{4-x}{2-\sqrt{|x|}} - 4\right| \leq \frac{|x-4|}{2} < \frac{\delta}{2} \leq \varepsilon.$$

Therefore

$$\lim_{x\to 4} \frac{4-x}{2-\sqrt{|x|}} = 4.$$

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Classwork

Use the ε-δ definition of limit to evaluate the following limits.

(a) $\lim\limits_{x\to 4} f(x)$, where $f : \mathbb{R} \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} 2x - 5 & \text{if } x \in \mathbb{R} \text{ is rational} \\ 15 - 3x & \text{if } x \in \mathbb{R} \text{ is irrational.} \end{cases}$$

(b) $\lim\limits_{x\to 2} \dfrac{4x-3}{x^2-5}$.