THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050C Mathematical Analysis I

Tutorial 6 (March 10)

Divergence Criteria. *If a sequence $X = (x_n)$ of real numbers has either of the following properties, then X is divergent.*

(i) X has two convergent subsequences $X' = (x_{n_k})$ and $X'' = (x_{r_k})$ whose limits are not equal.

(ii) X is not bounded.

Example 1. (a) Show that the sequence $X \coloneqq ((-1)^n)$ is divergent.

(b) Show that the sequence $Y = (y_n) \coloneqq \left(1, \frac{1}{2}, 3, \frac{1}{4}, \dots\right)$ is divergent.

(c) Show that the sequence $S \coloneqq (\sin n)$ is divergent.

Example 2. Suppose that $x_n \geq 0$ for all $n \in \mathbb{N}$ and that $\lim((-1)^n x_n)$ exists. Show that (x_n) converges.

Solution. Recall that if (y_n) converges, then $(|y_n|)$ also converges. Since $x_n \geq 0$ for all n, we have $x_n = |x_n| = |(-1)^n x_n|$. Hence the convergence of (x_n) follows from the convergence of $((-1)^n x_n)$. ◀

Definition. Let $X = (x_n)$ be a bounded sequence of real numbers. Let

$$\mathcal{L} = \{\ell \in \mathbb{R} : \exists \text{ subseq } (x_{n_k}) \text{ of } (x_n) \text{ s.t. } (x_{n_k}) \to \ell\}.$$

The **limit superior** and **limit inferior** of (x_n) are defined, respectively, as

$$\limsup(x_n) = \overline{\lim}(x_n) \coloneqq \sup \mathcal{L},$$
$$\liminf(x_n) = \underline{\lim}(x_n) \coloneqq \inf \mathcal{L}.$$

Theorem. *(a) Let $u_m \coloneqq \sup\{x_n : n \geq m\}$. Then (u_m) is decreasing and satisfies*

$$\limsup(x_n) = \lim(u_m) = \inf\{u_m : m \in \mathbb{N}\}.$$

(b) Let $v_m \coloneqq \inf\{x_n : n \geq m\}$. Then (v_m) is increasing and satisfies

$$\liminf(x_n) = \lim(v_m) = \sup\{v_m : m \in \mathbb{N}\}.$$

Example 3. Alternate the terms of the sequences $(1 + 1/n)$ and $(-1/n)$ to obtain the sequence (x_n) given by

$$(2, -1, 3/2, -1/2, 4/3, -1/3, 5/4, -1/4, \dots).$$

Determine the values of $\limsup(x_n)$ and $\liminf(x_n)$. Also find $\sup\{x_n\}$ and $\inf\{x_n\}$.

Solution. Observe that

$$v_m := \inf\{x_n : n \geq m\} = \begin{cases} x_{m+1} & \text{if } m \text{ is odd} \\ x_m & \text{if } m \text{ is even} \end{cases} = \begin{cases} -\frac{1}{(m+1)/2} & \text{if } m \text{ is odd} \\ -\frac{1}{m/2} & \text{if } m \text{ is even.} \end{cases}$$

Hence $\liminf(x_n) = \lim(v_m) = 0$.

Since $x_2 = -1$ is a lower bound of $\{x_n\}$, we have $\inf\{x_n\} = -1$.

Observe that

$$u_m := \sup\{x_n : n \geq m\} = \begin{cases} x_m & \text{if } m \text{ is odd} \\ x_{m+1} & \text{if } m \text{ is even} \end{cases} = \begin{cases} 1 + \frac{1}{(m+1)/2} & \text{if } m \text{ is odd} \\ 1 + \frac{1}{(m+2)/2} & \text{if } m \text{ is even.} \end{cases}$$

Hence $\limsup(x_n) = \lim(u_m) = 1$.

Since $x_1 = 2$ is an upper bound of $\{x_n\}$, we have $\sup\{x_n\} = 2$. ◀

Classwork

1. Prove that a bounded divergent sequence has two subsequences converging to different limits.

2. Show that if (x_n) is a bounded sequence, then (x_n) converges if and only if $\limsup(x_n) = \liminf(x_n)$.