THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050C Mathematical Analysis I

Tutorial 2 (January 27)

1 The Completeness Property of $\mathbb{R}$

The Completeness Property of $\mathbb{R}$. *Every nonempty set of real numbers that has an upper bound also has a supremum in $\mathbb{R}$.*

Example 1. (a) Let S be a nonempty subset of $\mathbb{R}$ that is bounded above, and let a be any real number in $\mathbb{R}$. Define the set $a + S := \{a + s : s \in S\}$. Show that

$$\sup(a + S) = a + \sup S.$$

(b) Let A and B be nonempty subsets of $\mathbb{R}$ that satisfy the property:

$$a \leq b \qquad \text{for all } a \in A \text{ and } b \in B.$$

Show that $\sup A \leq \inf B$.

Example 2. Suppose that f and g are real-valued functions with common domain $D \subseteq \mathbb{R}$. Assume that f and g are bounded (that is, $f(D)$ and $g(D)$ are bounded subsets of $\mathbb{R}$).

(a) If $f(x) \leq g(x)$ for all $x \in D$, show that $\sup f(D) \leq \sup g(D)$.

(b) If $f(x) \leq g(x)$ for all $x \in D$, is it true that $\sup f(D) \leq \inf g(D)$?

(c) If $f(x) \leq g(y)$ for all $x, y \in D$, show that $\sup f(D) \leq \inf g(D)$.

Solution. (c) Given $y \in D$, we have $f(x) \leq g(y)$ for all $x \in D$. So $g(y)$ is an upper bound for $f(D)$. Hence $\sup f(D) \leq g(y)$. Since the last inequality holds for all $y \in D$, we see that $\sup f(D)$ is a lower bound for $g(D)$. Therefore, we conclude that $\sup f(D) \leq \inf g(D)$. ◀

Classwork

1. Let S be a nonempty bounded subset of $\mathbb{R}$. Show that if $b < 0$,

$$\inf(bS) = b \sup S.$$

2. Let A and B be bounded nonempty subsets of $\mathbb{R}$, and let $A + B := \{a + b : a \in A, b \in B\}$. Prove that

$$\sup(A + B) = \sup A + \sup B \quad \text{and} \quad \inf(A + B) = \inf A + \inf B.$$