Optimization Theory

Tutorial 4

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Continuity of Convex Functions

Theorem

If $f : \Re^n \to \Re$ is convex, then it is continuous. More generally, if $f : \Re^n \to (-\infty, \infty]$ is a proper convex function, then f, restricted to $dom(f)$, is continuous over the relative interior of $dom(f)$.

Continuity of Convex Functions

Theorem

If C is closed interval of the real line, and $f : C \to \Re$ is closed and convex, then f is continuous over C.

Hyperplane

A **hyperplane** H in $\Re^n$ is a set of the form $\{x|a'x = b\}$ where a is a nonzero vector in $\Re^n$ and b is a scalar.
If $\bar{x} \in H$, then

$$H = x|a'x = a'\bar{x}\},$$

or

$$H = \bar{x} + \{x|a'x = 0\}.$$

H is an affine set that is parallel to the subspace $\{x|a'x = 0.\}$

$$\{x|a'x \geq b\}, \{x|a'x \leq b\}$$

are called the **closed halfspaces** associated with the hyperplane H.

$$\{x|a'x > b\}, \{x|a'x < b\}$$

are called the **open halfspaces** associated with the hyperplane H.

Supporting Hyperplane Theorem

Theorem

Let C be a nonempty convex subset of $\Re^n$ and let $\bar{x}$ be a vector in $\Re^n$. If $\bar{x}$ is not an interior point of C, there exists a hyperplane that passes through $\bar{x}$ and contains C in one of its closed halfplaces, i.e., there exists a vector $a \neq 0$ such that

$$a'\bar{x} \leq a'x, \forall x \in C.$$

Separating Hyperplane Theorem

Theorem

Let C_1 and C_2 be two nonempty convex subsets of $\Re^n$. If C_1 and C_2 are disjoint, there exists a hyperplane that separates them, i.e., there exists a vector $a \neq 0$ such that

$$a'x_1 \neq a'x_2, \forall x_1 \in C_1, \forall x_2 \in C_2.$$

Strict Separation Theorem

Theorem

Let C_1 and C_2 be two disjoint nonempty convex sets. There exists a hyperplane that strictly separates C_1 and C_2 under any one of the following five conditions:

(1) $C_2 - C_1$ *is closed.*

(2) C_1 *is closed and* C_2 *is compact.*

(3) C_1 *and* C_2 *are polyhedral.*

(4) C_1 *and* C_2 *are closed, and*

$$R_{C_1} \cap R_{C_2} = L_{C_1} \cap L_{C_2},$$

where R_{C_i} *and* L_{C_i} *denotes the recession cone and the lineality space of* C_i, $i = 1, 2$.

(5) C_1 *is closed,* C_2 *is polyhedral, and* $R_{C_1} \cap R_{C_2} \subset L_{C_1}$.

Corollary of Strict Separation Theorem

Theorem

The closure of the convex hull of a set C is the intersection of the closed halfspaces that contain C. In particular, a closed convex set is the intersection of the closed halfspaces that contain it.

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EX 1

Give an example of two closed convex sets that are disjoint but cannot be strictly separated.

EX 1

Give an example of two closed convex sets that are disjoint but cannot be strictly separated.

solution:

Take $C = \{x \in R^2 | x_2 \neq 0\}$
$D = \{x \in R^2_+ | x_1 x_2 \geq 1\}$.

Ex 2

Express the closed convex set $\{x \in \Re^2_+ | x_1 x_2 \geq 1\}$ as an intersection of halfspaces.

Solution. The set is the intersection of all supporting halfspaces at points in its boundary, which is given by $\{x \in \mathbf{R}^2_+ \mid x_1 x_2 = 1\}$. The supporting hyperplane at $x = (t, 1/t)$ is given by

$$x_1/t^2 + x_2 = 2/t,$$

so we can express the set as

$$\bigcap_{t>0} \{x \in \mathbf{R}^2 \mid x_1/t^2 + x_2 \geq 2/t\}.$$

Ex 2

Let $C = \{x \in \Re^n | \quad ||x||_\infty \leq 1\}$, the $l_\infty-$norm unit ball in $\Re^n$, and let $\hat{x}$ be a point in the boundary of C. Identity the supporting hyperplanes of C at $\hat{x}$ explicitly.

Solution. $s^T x \geq s^T \hat{x}$ for all $x \in C$ if and only if

$$\begin{array}{ll} s_i < 0 & \hat{x}_i = 1 \\ s_i > 0 & \hat{x}_i = -1 \\ s_i = 0 & -1 < \hat{x}_i < 1. \end{array}$$

Ex 3

Let $f : \Re^n \rightarrow \Re$ be a convex function and X be a bounded set in $\Re^n$. Show that f is Lipschitz continuous over X, i.e., there exists a positive scalar L such that

$$|f(x) - f(y)| \leq L||x - y||, \forall x, y \in X.$$

Let ϵ be a positive scalar and let C_ϵ be the set given by

$$C_\epsilon = \{z \mid \|z - x\| \le \epsilon, \text{ for some } x \in \text{cl}(X)\}.$$

We claim that the set C_ϵ is compact. Indeed, since X is bounded, so is its closure, which implies that $\|z\| \le \max_{x \in \text{cl}(X)} \|x\| + \epsilon$ for all $z \in C_\epsilon$, showing that C_ϵ is bounded. To show the closedness of C_ϵ, let $\{z_k\}$ be a sequence in C_ϵ converging to some z. By the definition of C_ϵ, there is a corresponding sequence $\{x_k\}$ in $\text{cl}(X)$ such that

$$\|z_k - x_k\| \le \epsilon, \qquad \forall\, k. \tag{2.1}$$

Because $\text{cl}(X)$ is compact, $\{x_k\}$ has a subsequence converging to some $x \in \text{cl}(X)$. Without loss of generality, we may assume that $\{x_k\}$ converges to $x \in \text{cl}(X)$. By taking the limit in Eq. (2.1) as $k \to \infty$, we obtain $\|z - x\| \le \epsilon$ with $x \in \text{cl}(X)$, showing that $z \in C_\epsilon$. Hence, C_ϵ is closed.

Ex 3

We now show that f has the Lipschitz property over X. Let x and y be two distinct points in X. Then, by the definition of C_ϵ, the point

$$z = y + \frac{\epsilon}{\|y - x\|}(y - x)$$

is in C_ϵ. Thus

$$y = \frac{\|y - x\|}{\|y - x\| + \epsilon} z + \frac{\epsilon}{\|y - x\| + \epsilon} x,$$

showing that y is a convex combination of $z \in C_\epsilon$ and $x \in C_\epsilon$. By convexity of f, we have

$$f(y) \leq \frac{\|y - x\|}{\|y - x\| + \epsilon} f(z) + \frac{\epsilon}{\|y - x\| + \epsilon} f(x),$$

implying that

$$f(y) - f(x) \leq \frac{\|y - x\|}{\|y - x\| + \epsilon} \big(f(z) - f(x)\big) \leq \frac{\|y - x\|}{\epsilon} \left(\max_{u \in C_\epsilon} f(u) - \min_{v \in C_\epsilon} f(v) \right),$$

Ex 3

where in the last inequality we use Weierstrass' theorem (f is continuous over $\Re^n$ by Prop. 1.4.6 and C_ϵ is compact). By switching the roles of x and y, we similarly obtain

$$f(x) - f(y) \leq \frac{\|x-y\|}{\epsilon} \left(\max_{u \in C_\epsilon} f(u) - \min_{v \in C_\epsilon} f(v) \right),$$

which combined with the preceding relation yields $\big|f(x) - f(y)\big| \leq L\|x-y\|$, where $L = \big(\max_{u \in C_\epsilon} f(u) - \min_{v \in C_\epsilon} f(v)\big)/\epsilon$.

Ex 4

Let C_1 and C_2 be nonempty convex subset of $\Re^n$, and let B denote the unit ball in $\Re^n$, $B = \{||x|| \leq 1\}$. A hyperplane H is said to separate strongly C_1 and C_2 and if there exists an $\epsilon > 0$ such that $C_1 + \epsilon B$ is contained in one of the open halfspaces associated with H and $C_2 + \epsilon B$ is contained in the other. Show that:

(a) The following three conditions are equivalent.

  (i) There exists a hyperplane separating strongly C_1 and C_2.

  (ii) There exists a vector $a \in \Re^n$ such that $inf_{x \in C_1} a'x > sup_{x \in C_2} a'x$.

  (iii) $inf_{x_1 \in C_1, x_2 \in C_2} ||x_1 - x_2|| > 0, i.e., 0 \neq cl(C_2 - C_1)$.

(b) If C_1 and C_2 are disjoint, any one of the five conditions for strict separation, implies that C_1 and C_2 can be strongly separated.

Ex 4

(a) We first show that (i) implies (ii). Suppose that C_1 and C_2 can be separated strongly. By definition, this implies that for some nonzero vector $a \in \Re^n$, $b \in \Re$, and $\epsilon > 0$, we have

$$C_1 + \epsilon B \subset \{x \mid a'x > b\},$$

$$C_2 + \epsilon B \subset \{x \mid a'x < b\},$$

where B denotes the closed unit ball. Since $a \neq 0$, we also have

$$\inf\{a'y \mid y \in B\} < 0, \qquad \sup\{a'y \mid y \in B\} > 0.$$

Therefore, it follows from the preceding relations that

$$b \leq \inf\{a'x + \epsilon a'y \mid x \in C_1, y \in B\} < \inf\{a'x \mid x \in C_1\},$$

$$b \geq \sup\{a'x + \epsilon a'y \mid x \in C_2, y \in B\} > \sup\{a'x \mid x \in C_2\}.$$

Thus, there exists a vector $a \in \Re^n$ such that

$$\inf_{x \in C_1} a'x > \sup_{x \in C_2} a'x,$$

Ex 4

Next, we show that (ii) implies (iii). Suppose that (ii) holds, i.e., there exists some vector $a \in \Re^n$ such that

$$\inf_{x \in C_1} a'x > \sup_{x \in C_2} a'x, \tag{2.15}$$

Using the Schwartz inequality, we see that

$$\begin{aligned} 0 &< \inf_{x \in C_1} a'x - \sup_{x \in C_2} a'x \\ &= \inf_{x_1 \in C_1,\ x_2 \in C_2} a'(x_1 - x_2), \\ &\leq \inf_{x_1 \in C_1,\ x_2 \in C_2} \|a\| \|x_1 - x_2\|. \end{aligned}$$

It follows that

$$\inf_{x_1 \in C_1, x_2 \in C_2} \|x_1 - x_2\| > 0,$$

thus proving (iii).

Ex 4

Finally, we show that (iii) implies (i). If (iii) holds, we have for some $\epsilon > 0$,

$$\inf_{x_1 \in C_1,\, x_2 \in C_2} \|x_1 - x_2\| > 2\epsilon > 0.$$

From this we obtain for all $x_1 \in C_1$, all $x_2 \in C_2$, and for all y_1, y_2 with $\|y_1\| \leq \epsilon$, $\|y_2\| \leq \epsilon$,

$$\|(x_1 + y_1) - (x_2 + y_2)\| \geq \|x_1 - x_2\| - \|y_1\| - \|y_2\| > 0,$$

which implies that $0 \notin (C_1 + \epsilon B) - (C_2 + \epsilon B)$. Therefore, the convex sets $C_1 + \epsilon B$ and $C_2 + \epsilon B$ are disjoint. By the Separating Hyperplane Theorem, we see that $C_1 + \epsilon B$ and $C_2 + \epsilon B$ can be separated, i.e., $C_1 + \epsilon B$ and $C_2 + \epsilon B$ lie in opposite closed halfspaces associated with the hyperplane that separates them. Then, the sets $C_1 + (\epsilon/2)B$ and $C_2 + (\epsilon/2)B$ lie in opposite open halfspaces, which by definition implies that C_1 and C_2 can be separated strongly.

Ex 4

(b) Since C_1 and C_2 are disjoint, we have $0 \notin (C_1 - C_2)$. Any one of conditions (2)-(5) of Prop. 2.4.3 imply condition (1) of that proposition (see the discussion in the proof of Prop. 2.4.3), which states that the set $C_1 - C_2$ is closed, i.e.,

$$\mathrm{cl}(C_1 - C_2) = C_1 - C_2.$$

Hence, we have $0 \notin \mathrm{cl}(C_1 - C_2)$, which implies that

$$\inf_{x_1 \in C_1,\, x_2 \in C_2} \|x_1 - x_2\| > 0.$$

From part (a), it follows that there exists a hyperplane separating C_1 and C_2 strongly.