MATH4230 - Optimization Theory
2017-2018

Mid-term (60 minutes)

1. (40marks)

   a. Let C be a nonempty subset of $\mathbb{R}^n$, and let λ_1 and λ_2 be positive scalars. Show that if C is convex, then $(\lambda_1 + \lambda_2)C = \lambda_1 C + \lambda_2 C$. Show by example that this need not be true when C is not convex.

   b. Show that a subset C is a convex cone if and only if it is closed under addition and positive scalar multiplication, i.e., $C + C \subseteq C$, and $\gamma C \subseteq C$ for all $\gamma > 0$.

   Solution

   a. We always have $(\lambda_1 + \lambda_2)C \subset \lambda_1 C + \lambda_2 C$, even if C is not convex. To show the reverse inclusion assuming C is convex, note that a vector x in $\lambda_1 C + \lambda_2 C$ is of the form $x = \lambda_1 x_1 + \lambda_2 x_2$, where $x_1, x_2 \in C$. By convexity of C, we have

   $$\frac{\lambda_1}{\lambda_1 + \lambda_2} x_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} x_2 \in C,$$

   and it follows that

   $$x = \lambda_1 x_1 + \lambda_2 x_2 \in (\lambda_1 + \lambda_2)C,$$

   so $\lambda_1 C + \lambda_2 C \subset (\lambda_1 + \lambda_2)C$.

   For a counterexample C is not convex, let C be a set in R^n consisting of two vectors, 0 and $x \neq 0$, and let $\lambda_1 = \lambda_2 = 1$. Then C is not convex and $(\lambda_1 + \lambda_2)C = 2C = \{0, 2x\}$, while $\lambda_1 C + \lambda_2 C = C + C = \{0, x, 2x\}$, showing that $(\lambda_1 + \lambda_2)C \neq \lambda_1 C + \lambda_2 C$.

   b. Let C be a convex cone. Then $\gamma C \subset C$, for all $\gamma > 0$, by the definition of cone. Furthermore, by convexity of C, for all $x, y \in C$, we have $z \in C$, where $z = \frac{1}{2}(x + y)$. Hence $(x + y) = 2z \in C$, since C is a cone, and it follows that $C + C \subset C$.

   Conversely, assume that $C + C \subset C$ and $\gamma C \subset C$. Then C is a cone. Furthermore, if $x, y \in C$ and $\alpha \in (0, 1)$, we have $\alpha x \in C$ and $(1 - \alpha)y \in C$ and $\alpha x + (1 - \alpha)y \in C$. Hence C is convex.

2. (40marks) Prove the following statements:

   a. If X_1 and X_2 are convex sets that can be separated by a hyperplane, and X_1 is open, then X_2 and X_2 are disjoint.

   b. If $f : \mathbb{R}^n \mapsto \mathbb{R}$ is a convex function that is bounded in the sense that for some $\gamma > 0$, $|f(x)| \leq \gamma$ for all $x \in \mathbb{R}^n$, then the problem

   $$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathbb{R}^n. \end{array}$$

   has a solution.

   Solution

   a. Since there exist a hyperplane separates them, that is, $\exists a$ and b such that

   $$a^T x_1 \leq b \leq a^T x_2, \; x_1 \in X_1, x_2 \in X_2.$$

   Suppose $X_1 \cap X_2 \neq \emptyset$. , so $x^* \in X_1 \cap X_2$, we have $a^T x^* = b$.

Since $x^* \in X_1$, which is open, we get $x^* + \epsilon\frac{a}{\|a\|} \in X_1$, where $\epsilon > 0$. then

$$a^T(x^* + \epsilon\frac{a}{\|a\|}) = b + \epsilon\|a\| > b$$

So we get the contradiction as $a^T x_1 \leq b, \forall x_1 \in X_1$.
Therefore We get $X_1 \cap X_2 = \emptyset$

b. Suppose f is not constant, i.e., $\exists x, y \in R^N : f(x) > f(y)$. Since f is convex, we have:

$$f(x) \leq \lambda f(\frac{x - (1-\lambda)y}{\lambda}) + (1-\lambda)f(y), \ \forall\lambda \in (0,1).$$

Hence, $\frac{f(x)-(1-\lambda)f(y)}{\lambda} \leq f(\frac{x-(1-\lambda)}{\lambda})$. Since $f(x) > f(y)$, $\frac{f(x)-(1-\lambda)f(y)}{\lambda} = \frac{f(x)-f(y)}{\lambda} + f(y) \rightarrow \infty$ as $\lambda \rightarrow 0^+$. Hence f is not bounded which is contradicted with $|f(x)| \leq \gamma, \ \forall x$.
Therefore, f is constant and the minimization has a solution.

3. **a.** (20marks) Let C be a nonempty convex cone. Show that $cl(C)$ and $ri(C)$ are also convex cones.

b. (Optional 5marks) Let $C = \text{cone}(\{x_1, ..., x_m\})$. Show that

$$ri(C) = \{\sum_{i=1}^{m} a_i x_i | a_i > 0, i = 1, \ldots, m\}.$$

Solution

a. Let $x \in cl(C)$ and let α be a positive scalar. Then, there exists a sequence $\{x_k\} \in C$ such that $x_k \rightarrow x$, and since C is a cone ,$\alpha x_k \in C$ for all k. Furthermore, $\alpha x_k \rightarrow \alpha x$, implying that $\alpha x \in cl(C)$. Hence, $cl(C)$ is a cone, and it also convex since the closure of a convex set is convex.

By Nonemptiness of Relative Interior Theorem, the relative interior of a nonempty convex set is convex. To show that $ri(C)$ is a cone, let $x \in ri(C)$. Then, $x \in C$ and since C is a cone, $\alpha x \in C$ for all $\alpha > 0$. By the Line Segment Principle, all the points on the line segment connecting x and αx, except possibly αx, belong to $ri(C)$,

$$\text{i.e.} \quad \beta x \in ri(C), \beta \in (\alpha, 1] \text{ or } [1, \alpha).$$

Since this is true for every $\alpha > 0$, it follows that $\beta x \in ri(C)$ for all $\beta > 0$, then showing that $ri(C)$ is a cone.

b. Consider the linear transformation A that maps $(\alpha_1, \ldots, \alpha_m) \in \Re^m$ into $\sum_{i=1}^{i=m} \alpha_i x_i \in \Re^n$. Note that C is the image of the nonempty convex set

$$\{(\alpha_1, \ldots, \alpha_m) | \alpha_1 \geq 0, \ldots, \alpha_m \geq 0\}$$

under A. Therefore, we have

$$\begin{aligned} ri(C) &= ri(A \cdot \{(\alpha_1, \ldots, \alpha_m \geq 0)\}) \\ &= A \cdot ri((\alpha_1, \ldots, \alpha_m \geq 0)) \quad \text{(prop.1.3.6)} \\ &= A \cdot \{(\alpha_1, \ldots, \alpha_m \geq 0)\} \\ &= \left\{ \sum_{i=1}^{i=m} \alpha_i x_i | \alpha_1 > 0, \ldots, \alpha_m > 0 \right\} \quad \text{(prop.1.3.6)}. \end{aligned}$$

Alternative solution of b:
WLOG, assume $x_1, x_2, \ldots, x_m$ are linearly independent. C is a cone, then

$$C = \left\{ \sum_{i=1}^{i=m} \alpha_i x_i | \alpha_1 \geq 0, \ldots, \alpha_m \geq 0 \right\}. \qquad (*)$$

Denote $A = \left\{ \sum_{i=1}^{i=m} \alpha_i x_i | \alpha_1 > 0, \ldots, \alpha_m > 0 \right\}$. We will prove $A = ri(C)$.
Obviously, A is open. $\forall x \in A$, there exists a ball $B(x, r_x)$ such that $B(x, r_x) \subset A \subset C$. And $A \subset aff(C)$. Thus $(B(x, r_x) \cap aff(C)) \subset C$. Hence, $x \in ri(C)$.
On the other hand, $\forall x \in ri(C), x \in C$. Then, $x = \sum_{i=1}^{i=m} \alpha_i x_i, \alpha_i \geq 0$. It suffices to prove that $\alpha_i \neq 0$. Otherwise, WLOG, suppose $x = \sum_{i \neq k} \alpha_i x_i$. Obviously, $\hat{x} = \sum_{i=1}^{i=m} \alpha_i x_i \in C, \alpha_k > 0$. By Prolongation Lemma, there exist $\gamma > 0$ such that $x + \gamma(x - \hat{x}) = \sum_{i \neq k} \alpha_i x_i + \gamma(-\alpha_k x_k) \in C$. $-\gamma \alpha_k x < 0$, it contradicts with $(*)$. Hence, $x = \sum_{i=1}^{i=m} \alpha_i x_i, \alpha_i > 0$.