Math4230 Exercise 7 Solution

1. (a) $f^*(y) = \begin{cases} -1 - \log(-y) & \text{if } y < 0 \\ \infty & \text{otherwise} \end{cases}$

   (b) $f^*(y) = \frac{1}{2} y^T Q^{-1} y$

2. (a) $f_1^*(y) = g^*(y - a) - b$;

   (b) $f_2^*(y) = b^T y + g^*(y)$.

3. For $x \neq 0$, f is differentiable with $\nabla f(x) = \frac{x}{||x||}$.
   Hence $\partial f(x) = \{x/||x||\}$.
   For $x = 0$, if $f(y) \geq f(0) + \langle g, y \rangle$ for all y, then

   $$||y|| \geq \langle g, y \rangle, \ \forall y.$$

   Let $y = g$, then $||g|| \leq 1$, so $\partial f(0) \subset \{g | \ ||g|| \leq 1\}$.
   Conversely, suppose $||g|| \leq 1$, then

   $$\langle g, y \rangle \leq ||g|| ||y|| \leq ||y||$$

   Hence, $f(0) + \langle g, y \rangle \leq f(y)$.
   Therefore, $\{g | \ ||g|| \leq 1\} \subset \partial f(0)$.

4. Since $g_x \in \partial f(x)$,

   $$f(y) \geq f(x) + \langle g_x, y - x \rangle$$

   Similarly, since $g_y \in \partial f(y)$,

   $$f(x) \geq f(y) + \langle g_y, x - y \rangle$$

   Adding the two inequalities, we get

   $$\langle g_x - g_y, x - y \rangle \geq 0$$