Math4230 Exercise 6 Solution

1. (a) $x - \frac{\langle a,x\rangle - b}{\|a\|^2} a$

   (b) $\max\{x, 0\}$

2.
$$\begin{aligned}
&\|P_C(x_1) - P_C(x_2)\|^2 \\
&= \langle P_C(x_1) - x_2, P_C(x_1) - P_C(x_2)\rangle + \langle x_2 - P_C(x_2), P_C(x_1) - P_C(x_2)\rangle \\
&\leq \langle P_C(x_1) - x_2, P_C(x_1) - P_C(x_2)\rangle \\
&= \langle P_C(x_1) - x_1, P_C(x_1) - P_C(x_2)\rangle + \langle x_1 - x_2, P_C(x_1) - P_C(x_2)\rangle \\
&\leq \langle x_1 - x_2, P_C(x_1) - P_C(x_2)\rangle
\end{aligned}$$

3. Since C is closed and $\overline{x} \notin C$, C and $\overline{x}$ can be strictly separated by a hyperplane. Hence for some nonzero $a \in \mathbb{R}^n$ we have,
$$\langle a, x\rangle < \langle a, \overline{x}\rangle,\ \forall x \in C.$$
Suppose $\langle a, x'\rangle > 0$ for some $x' \in C$. Since C is a cone, $\lambda x \in C\ \forall \lambda > 0$. By choosing a large λ, we get a contradiction, since $\langle a, \lambda x'\rangle > \langle a, \overline{x}\rangle$. Hence $\langle a, x\rangle \leq 0,\ \forall x \in C$.
Since C is a closed cone, $0 \in C$. We must have $\langle a, \overline{x}\rangle > 0$.

4. Since V is closed and $\overline{x} \notin V$, V and $\overline{x}$ can be strictly separated by a hyperplane. Hence for some nonzero $a \in \mathbb{R}^n$ we have,
$$\langle a, x\rangle < \langle a, \overline{x}\rangle,\ \forall x \in V.$$
Suppose $\langle a, x'\rangle \neq 0$ for some $x' \in V$. Since V is a subspace, $\lambda x \in V$ for all λ. We can choose λ such that $\langle a, \lambda x'\rangle > \langle a, \overline{x}\rangle$. Hence, we get a contradiction. Since V is a subspace, $0 \in V$. We must have $\langle a, \overline{x}\rangle > 0$.

5. When $x > 0, y < 0$, then $z = (x, y) \in \text{int } C$, hence $N(z; C) = \{0\}$.
When $x = 0, y < 0$, then $N(z; C) = \{v_1, v_2 \mid v_1 \leq 0, v_2 = 0\}$.
When $x > 0, y = 0$, then $N(z; C) = \{v_1, v_2 \mid v_1 = 0, v_2 \geq 0\}$.
When $x = 0, y = 0$, then $N(z; C) = \{v_1, v_2 \mid v_1 \leq 0, v_2 \geq 0\}$.