Math4230 Exercise 6

1. Find the formula of $P_C(x)$ for the following sets C:

   (a) $C = \{x \in \mathbb{R}^n |\ \langle a, x\rangle = b\}$, where $0 \neq a \in \mathbb{R}^n$, $b \in \mathbb{R}$.
   (b) $C = \mathbb{R}^n_+ := \{x \in \mathbb{R}^n |\ x_i \geq 0,\ \forall i\}$.

2. Let C be a nonempty closed convex set. Show that for all $x_1, x_2 \in \mathbb{R}^n$,
$$\|P_C(x_1) - P_C(x_2)\|^2 \leq \langle P_C(x_1) - P_C(x_2), x_1 - x_2\rangle$$

3. Let $C \subseteq \mathbb{R}^n$ be a nonempty closed, convex cone. Suppose $\overline{x} \notin C$. Show that there exists $0 \neq a \in \mathbb{R}^n$ such that
$$\langle a, x\rangle \leq 0\ \forall x \in C,\ \text{and}\ \langle a, \overline{x}\rangle > 0$$

4. Let V be a non trivial subspace of $\mathbb{R}^n$. Suppose $\overline{x} \notin V$. Show that there exists $0 \neq a \in \mathbb{R}^n$ such that
$$\langle a, x\rangle = 0\ \forall x \in V,\ \text{and}\ \langle a, \overline{x}\rangle > 0$$

5. Let $C = \{(x, y) \mid x \geq 0, y \leq 0\}$. Compute the normal cone $N(\bar{x}; C)$ of C at every $\bar{x} \in C$.