2 Subdifferential Calculus

2.1 Convex Separation

The separating theorems are of fundamental importance in convex analysis and optimization. This section provides some of the useful results.

Definition:(Hyperplane Separation) Two sets C_1, C_2 are said to be separated by a hyperplane if there exists $a \neq 0$ such that

$$\sup_{x \in C_1} \langle a, x \rangle \leq \inf_{y \in C_2} \langle a, y \rangle$$

C_1, C_2 are said to be strictly separated if there exists $a \neq 0$ such that

$$\sup_{x \in C_1} \langle a, x \rangle < \inf_{y \in C_2} \langle a, y \rangle$$

If x is a relative boundary point of C, a hyperplane that separates C and $\{x\}$ is called a supporting hyperplane at x.

We will focus on the separation of two convex sets. To proof the existence of such separation, we start with two lemmas.
Lemma: Let C be an nonempty, closed convex set and $\overline{x} \notin C$. Then there exists nonzero a such that

$$\sup_{x \in C} \langle a, x \rangle < \langle a, \overline{x} \rangle$$

Proof. Let $w = P_C(\overline{x})$ (which exists by the projection property). Then

$$\langle \overline{x} - w, x \rangle \leq \langle \overline{x} - w, w \rangle \text{ for all } x \in C.$$

Let $a = \overline{x} - w \neq 0$, then

$$\langle a, x \rangle \leq \langle a, w \rangle = \langle a, \overline{x} \rangle - ||\overline{x} - w||^2 < \langle a, \overline{x} \rangle$$

□

Lemma: Let C be a nonempty, convex subset of $\mathbb{R}^n$ with $x \in \overline{C} \backslash \text{ri}(C)$. Then there exists $\{x_k\}$ such that $x_k \to x$ while $x_k \notin \overline{C}$ for all k.

Proof. Since ri(C) is nonempty, pick $x_0 \in$ri(C).
Let $x_k = \frac{k+1}{k} x - \frac{x_0}{k}$.
Clearly, $x_k \to x$. It remains to show that $x_k \notin \overline{C}$. Suppose otherwise, then by the Line Segment property,

$$x = \frac{1}{k+1} x_0 + \frac{k}{k+1} \left(\frac{k+1}{k} x - \frac{x_0}{k}\right) \in \text{ri}(C)$$

This is a contradiction. Hence $x_k \notin \overline{C}$ for all k.

□

Theorem:(Supporting Hyperplane Theorem) Let C be a nonempty, convex set. Suppose $\overline{x} \in \text{rel } \partial C = \overline{C} \backslash \text{ri}(C)$. Then there exists $a \neq 0$ such that

$$\sup_{x \in \overline{C}} \langle a, x \rangle \leq \langle a, \overline{x} \rangle$$

Proof. Since $\overline{x} \in \text{rel } \partial C$. Then there exists $x_k \notin \overline{C}$ with $x_k \to \overline{x}$.
By lemma, there exists $a_k \neq 0$ such that

$$\sup_{x \in \overline{C}} \langle a_k, x \rangle < \langle a_k, x_k \rangle$$

By dividing $\|a_k\|$, we may assume $\{a_k\}$ is bounded.
Since $\{a_k\}$ is bounded, it has a converging subsequence.
Without loss of generality (considering the subsequence), we may assume that $a_k \to a \neq 0$
Taking the limit, we have for all $x \in \overline{C}$

$$\langle a, x \rangle \leq \langle a, \overline{x} \rangle$$

□

Theorem:(Separating Hyperplane Theorem) Let C_1, C_2 be two convex sets. Suppose $C_1 \cap C_2 = \emptyset$. Then there exists a hyperplane that separates C_1 and C_2.

Proof. Consider $C := C_1 - C_2$. Since $C_1 \cap C_2 = \emptyset$, $0 \notin C$.
There are two cases:
Case (1): $0 \in \overline{C}$.
By the supporting hyperplane theorem, there exists $a \neq 0$ such that

$$\langle a, x \rangle \leq \langle a, 0 \rangle = 0, \text{ for all } x \in C$$

That is

$$\langle a, x_1 \rangle \leq \langle a, x_2 \rangle$$

Case (2): $0 \notin \overline{C}$
The result follows directly from the previous lemma. □

In order to get strict separation, we need more assumptions.

Theorem:(Strict Hyperplane Separation) Let C_1, C_2 be nonempty, closed convex sets with $C_1 \cap C_2 = \emptyset$. Suppose at least one of the two sets is also bounded. Then there exists $a \neq 0$ such that

$$\sup_{x_1 \in C_1} \langle a, x_1 \rangle < \inf_{x_2 \in C_2} \langle a, x_2 \rangle$$

Proof. Let $C := C_1 - C_2$. Then C is a nonempty, closed convex set with $0 \notin C$. Then there exists $a \neq 0$ such that

$$\gamma := \sup_{x \in C} \langle a, x \rangle < 0$$

Then for all $x_1 \in C_1$, $x_2 \in C_2$, we have $\langle a, x_1 \rangle \leq \gamma + \langle a, x_2 \rangle$. Then

$$\sup_{x_1 \in C_1} \langle a, x_1 \rangle \leq \gamma + \inf_{x_2 \in C_2} \langle a, x_2 \rangle < \inf_{x_2 \in C_2} \langle a, x_2 \rangle$$

□

2.2 Lipschitz Continuity

In this section, we focus on the Lipschitz continuity of convex functions.
First, we start with some lemmas.

Lemma: Let $\{e_1, ..., e_n\}$ denote the standard basis of $\mathbb{R}^n$. Let $A := \{x \pm \epsilon e_i\}$ Then the following holds:

1. $x + \delta e_i \in \text{conv}(A)$ for $|\delta| \leq \epsilon$
2. $B(x; \epsilon/n) \subset \text{conv}(A)$

Proof. 1. Since $|\delta| \leq \epsilon$, there exists λ such that $\delta = \lambda(-\epsilon) + (1-\lambda)\epsilon$. Then,

$$x + \delta e_i = \lambda(x - \epsilon e_i) + (1-\lambda)(x + \epsilon e_i) \in \text{conv}(A)$$

2. Let $y \in B(x; \epsilon/n)$. Then $y = x + \frac{\epsilon}{n} u$, where $||u|| \leq 1$. Write $u = \sum_{i=1}^n \lambda_i e_i$, then

$$|\lambda_i| \leq \sqrt{\sum_{i=1}^n \lambda_i^2} \leq 1$$

So

$$y = x + \frac{\epsilon}{n} u = x + \frac{\epsilon}{n} \sum_{i=1}^n \lambda_i e_i = \sum_{i=1}^n \frac{1}{n}(x + \epsilon \lambda_i e_i)$$

Since $x + \epsilon \lambda_i e_i \in \text{conv}(A)$, $y \in \text{conv}(A)$. Hence $B(x; \frac{\epsilon}{n}) \subseteq \text{conv}(A)$.

□

Lemma: If a convex function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is bounded above on $B(\overline{x}; \delta)$ for some $\overline{x} \in \text{dom} f$ and $\delta > 0$, then f is bounded on $B(\overline{x}, \delta)$.

Proof. Suppose $f(x) \leq M$ for all $x \in B(\overline{x}, \delta)$. Let $f(\overline{x}) = m$.
Suppose $x \in B(\overline{x}; \delta)$ Let $u := \overline{x} + (\overline{x} - x) = 2\overline{x} - x$. Then $u \in B(\overline{x}, \delta)$. We have

$$m = f(\overline{x}) = f(\frac{x+u}{2}) \leq \frac{1}{2} f(x) + \frac{1}{2} f(u)$$

Therefore, $f(x) \geq 2f(\overline{x}) - f(u) \geq 2m - M$. Hence f is bounded on $B(\overline{x}, \delta)$. □

Theorem: Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be convex with $\overline{x} \in \text{dom} f$. Suppose f is bounded on $B(\overline{x}, \delta)$ for some $\delta > 0$, then f is Lipschitz continuous on $B(\overline{x}; \frac{\delta}{2})$.

Proof. Let $x, y \in B(\overline{x}; \frac{\delta}{2})$ with $x \neq y$. Suppose $f \leq M$ on $B(\overline{x}; \delta)$. Let

$$u := x + \frac{\delta}{2||x-y||}(x-y)$$

then $u \in x + \frac{\delta}{2}B \subset x + \delta B$. Also

$$x = \frac{1}{\alpha+1}u + \frac{\alpha}{\alpha+1}y$$

where $\alpha = \frac{\delta}{2||x-y||}$. Then

$$\begin{aligned} f(x) - f(y) &\leq \frac{1}{\alpha+1}f(u) + \frac{\alpha}{\alpha+1}f(y) - f(y) \\ &= \frac{1}{\alpha+1}(f(u) - f(y)) \leq \frac{2M}{\alpha+1} \\ &= \frac{4M||x-y||}{\delta + 2||x-y||} \leq \frac{4M||x-y||}{\delta} \end{aligned}$$

□

Proposition: A convex function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is locally Lipschitz continuous on $\text{int}(\text{dom} f)$.

Proof. Let $\overline{x} \in \text{int}(\text{dom} f)$ and let $\epsilon > 0$ be such that $\overline{x} \pm \epsilon e_i \in \text{dom} f$ for all i. Let $A := \{\overline{x} \pm \epsilon e_i\}$. Then $B(\overline{x}; \frac{\epsilon}{n}) \subseteq \text{conv}(A)$. Let $M := \max\{f(a) |\ a \in A\}$. Pick $x \in B(\overline{x}; \frac{\epsilon}{n})$, then

$$x = \sum \lambda_i (\overline{x} + \epsilon e_i), \text{ with } \sum \lambda_i = 1$$

Hence

$$f(x) \leq \sum \lambda_i f(\overline{x} + \epsilon e_i) \leq M$$

Then f is bounded above on $B(\overline{x}; \frac{\epsilon}{n})$. Hence, by the previous theorem, f is Lipschitz continuous on $B(\overline{x}; \frac{\epsilon}{2n})$ □