1.4 Relative Interior

Consider $I = [0,1] \subset \mathbb{R}$. Then the interior of I is (0,1). However, if we consider I as a subset in $\mathbb{R}^2$, then the interior of I is empty. This motivates the following definition.

Definition:(Relative Interior) Let $C \subset \mathbb{R}^n$. We say that x is a *relative interior point* of C if $B(x;\epsilon) \cap \text{aff}(C) \subset C$, for some $\epsilon > 0$. The set of all relative interior point of C is called the *relative interior* of C, and is denoted by $\text{ri}(C)$. The *relative boundary* of C is equal to $\overline{C} \setminus \text{ri}(C)$.

Lemma: Let Δ_m be an m-simplex in $\mathbb{R}^n$ with $m \geq 1$. Then $\text{ri}(\Delta_m) \neq \emptyset$.

Proof. Let $x_0, ..., x_m$ be the vertices of Δ_m. Let

$$\overline{x} := \frac{1}{m+1} \sum_{i=0}^{m} x_i$$

Note that $V := \text{span}\{x_1 - x_0, ..., x_m - x_0\}$ is the m-dimensional subspace parallel to $\text{aff}(\Delta_m) = \text{aff}(\{x_0, ..., x_m\})$.
Hence for all $x \in V$, there exists unique λ_i such that

$$x = \sum_{i=1}^{m} \lambda_i (x_i - x_0)$$

Let $\lambda_0 := -\sum_{i=1}^{m} \lambda_i$, then $(\lambda_0, ..., \lambda_m) \in \mathbb{R}^{m+1}$ and

$$x = \sum_{i=0}^{m} \lambda_i x_i, \text{ with } \sum_{i=0}^{m} \lambda_i = 0$$

Let $L : V \to \mathbb{R}^{m+1}$ be the mapping that sends x to $(\lambda_0, ..., \lambda_m)$. It is easy to check that L is linear and thus continuous.
Hence there exists δ such that

$$||L(u)|| < \frac{1}{m+1} \text{ if } ||u|| < \delta$$

Let $x \in (\overline{x} + B(0,\delta)) \cap \text{aff}(\Delta_m)$ Then, $x = \overline{x} + u$, where $||u|| < \delta$.
Since $x, \overline{x} \in \text{aff}(\Delta_m)$ and $u = x - \overline{x}$, $u \in V$. Hence $||L(u)|| < \frac{1}{m+1}$.
Suppose $L(u) = (\mu_0, ..., \mu_m)$, then $u = \sum_{i=0}^{m} \mu_i x_i$ and $x = \sum_{i=0}^{m} (\frac{1}{m+1} + \mu_i) x_i$.
Since $\sum_{i=0}^{m} \mu_i = 0$, $\sum_{i=0}^{m} (\frac{1}{m+1} + \mu_i) = 1$. Therefore, $x \in \Delta_m$.
Thus $(\overline{x} + B(0;\delta)) \cap \text{aff}(\Delta_m) \subset \Delta_m$, so $\overline{x} \in \text{ri}(\Delta_m)$. □

Proposition: Let C be a nonempty convex set. Then $\text{ri}(C)$ is nonempty.

Proof. Let m be the dimension of C.
If $m = 0$, then C must be a singleton. Hence $\text{ri}(C) \neq \emptyset$.
Suppose $m \geq 1$. We first show that there exists $m + 1$ affinely independent

elements $x_0, ..., x_m \in C$.
Let $\{x_0, ..., x_k\}$ be a maximal affinely independent set in C.
Consider $K := \text{aff}(\{x_0, ..., x_k\})$. $K \subseteq \text{aff}(C)$ since $\{x_0, ..., x_k\} \subset C$.
Suppose $y \in C$ but $y \notin K$. Then, $\{x_0, ..., x_k, y\}$ is also affinely independent, which is a contradiction. Therefore $C \subseteq K$ and hence $\text{aff}(C) \subseteq K$. Then

$$k = \dim(K) = \dim(\text{aff}(C)) = m$$

Therefore, there exists $m + 1$ affinely independent elements $x_0, ..., x_m \in C$.
Let Δ_m be the m-simplex formed by $\{x_0, ..., x_m\}$. By above, $\text{aff}(\Delta_m) = \text{aff}(C)$.
Since $\text{ri}(\Delta_m)$ is not empty, it follows that $\text{ri}(C)$ is also nonempty. □

The following is the most fundamental result about relative interiors.
Proposition:(Line Segment Principle) Let C be a nonempty convex set. If $x \in \text{ri}(C)$, $\overline{x} \in \overline{C}$, then $\lambda x + (1 - \lambda)\overline{x} \in \text{ri}(C)$ for $\lambda \in (0, 1]$.

Proof. Fix $\lambda \in (0, 1]$. Consider $x_\lambda = \lambda x + (1 - \lambda)\overline{x}$.
Let L be the subspace parallel to $\text{aff}(C)$. Define $B_L(0, \epsilon) := \{z \in L | \; \|z\| < \epsilon\}$.
Since $\overline{x} \in \overline{C}$, for all $\epsilon > 0$, we have $\overline{x} \in C + B_L(0, \epsilon)$. Then

$$\begin{aligned} B(x_\lambda; \epsilon) \cap \text{aff}(C) &= \{\lambda x + (1 - \lambda)\overline{x}\} + B_L(0; \epsilon) \\ &\subset \{\lambda x\} + (1 - \lambda)C + (2 - \lambda)B_L(0; \epsilon) \\ &= (1 - \lambda)C + \lambda\left[x + B_L\left(0; \frac{2 - \lambda}{\lambda}\epsilon\right)\right] \end{aligned}$$

Since $x \in \text{ri}(C)$, $x + B_L\left(0; \frac{2-\lambda}{\lambda}\epsilon\right) \subset C$, for sufficiently small ϵ.
So $B(x_\lambda; \epsilon) \cap \text{aff}(C) \subset \lambda C + (1 - \lambda)C = C$ (since C is convex). Therefore, $x_\lambda \in \text{ri}(C)$.

□

Proposition:(Prolongation Lemma) Let C be a nonempty convex set. Then we have

$$x \in \text{ri}(C) \iff \forall \overline{x} \in C, \; \exists \gamma > 0 \text{ such that } x + \gamma(x - \overline{x}) \in C.$$

In other words, x is a relative interior point iff every line segment in C having x as one of the endpoints can be prolonged beyond x without leaving C.

Proof. Suppose the condition holds for x. Let $\overline{x} \in \text{ri}(C)$. If $x = \overline{x}$, then we are done. So assume $x \neq \overline{x}$. Then there exists $\gamma > 0$ such that $y = x + \gamma(x - \overline{x}) \in C$. Hence $x = \frac{1}{1+\gamma}y + \frac{\gamma}{1+\gamma}\overline{x}$. Since $\overline{x} \in \text{ri}(C)$, $y \in C$, by the line segment principle, we have $x \in \text{ri}(C)$. The other direction is clear from the fact that $x \in \text{ri}(C)$. □

Next, we introduce some calculus rules related to the relative interior of convex sets.

Proposition: Let C be a nonempty convex set. Then

(a) $\overline{C} = \overline{\mathrm{ri}(C)}$.

(b) $\mathrm{ri}(C) = \mathrm{ri}(\overline{C})$.

(c) Let D be another nonempty convex set. Then the following are equivalent:

   (i) C and D have the same relative interior.

   (ii) C and D have the same closure.

   (iii) $\mathrm{ri}(C) \subseteq D \subseteq \overline{C}$.

Proof. (a) $\overline{\mathrm{ri}(C)} \subset \overline{C}$ since $\mathrm{ri}(C) \subset C$. Conversely, suppose $x \in \overline{C}$.
Let $\overline{x} \in \mathrm{ri}(C)$. Consider $x_k = \frac{1}{k}\overline{x} + (1 - \frac{1}{k})x$. By the line segment principle, each $x_k \in \mathrm{ri}(C)$. Also, $x_k \to x$. Therefore, $x \in \overline{\mathrm{ri}(C)}$.

(b) Note that $\mathrm{aff}(C) = \mathrm{aff}(\overline{C})$. Then by the definition of relative interior, $\mathrm{ri}(C) \subseteq \mathrm{ri}(\overline{C})$. Now suppose $\overline{x} \in \mathrm{ri}(\overline{C})$, we will show that $\overline{x} \in \mathrm{ri}(C)$.
Pick $x \in \mathrm{ri}(C)$. We may assume $x \neq \overline{x}$.
Then by the prolongation lemma, there exists $\gamma > 0$ such that

$$\overline{x} + \gamma(\overline{x} - x) \in \overline{C}$$

Then by the line segment principle and the fact that $x \in \mathrm{ri}(C)$,

$$\overline{x} = \frac{\gamma}{\gamma+1}x + \frac{1}{\gamma+1}(\overline{x} + \gamma(\overline{x} - x)) \in \mathrm{ri}(C)$$

(c) Suppose $\mathrm{ri}(C)$=$\mathrm{ri}(D)$, then $\overline{\mathrm{ri}(C)} = \overline{\mathrm{ri}(D)}$. Hence $\overline{C} = \overline{D}$.
Suppose $\overline{C} = \overline{D}$, then $\mathrm{ri}(C)$=$\mathrm{ri}(\overline{C})$=$\mathrm{ri}(\overline{D})$=$\mathrm{ri}(D)$.
Therefore (i) and (ii) are equivalent.
Suppose $\overline{C} = \overline{D}$, then

$$\mathrm{ri}(C) = \mathrm{ri}(D) \subseteq D \subseteq \overline{D} = \overline{C}$$

Suppose $\mathrm{ri}(C) \subseteq D \subseteq \mathrm{cl}(C)$, then $\overline{\mathrm{ri}(C)} \subseteq \overline{D} \subseteq \overline{C}$.
Since $\overline{\mathrm{ri}(C)} = \overline{C}$, $\overline{\mathrm{ri}(C)} = \overline{D} = \overline{\mathrm{ri}(D)}$.
Hence $\overline{C} = \overline{D}$ and (ii),(iii) are equivalent.

□

Proposition: Let C_1 and C_2 be nonempty convex sets. We have

$$\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) \subseteq \mathrm{ri}(C_1 \cap C_2),\ \overline{C_1 \cap C_2} \subseteq \overline{C_1} \cap \overline{C_2}.$$

Furthermore, if $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) \neq \emptyset$, then

$$\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) = \mathrm{ri}(C_1 \cap C_2),\ \overline{C_1 \cap C_2} = \overline{C_1} \cap \overline{C_2}.$$

Proof. Let $x \in \mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)$, $y \in C_1 \cap C_2$. By the prolongation lemma, the line segment connecting x and y can be prolonged beyond x without leaving C_1 and C_2. Hence, by the prolongation lemma again, $x \in \mathrm{ri}(C_1 \cap C_2)$.
Since $C_1 \cap C_2 \subseteq \overline{C_1} \cap \overline{C_2}$, which is closed, we have $\overline{C_1 \cap C_2} \subseteq \overline{C_1} \cap \overline{C_2}$.
Now suppose $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) \neq \emptyset$ and let $x \in \mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)$ and $y \in \overline{C_1} \cap \overline{C_2}$. Consider $\alpha_k \to 0$ and $y_k = \alpha_k x + (1 - \alpha_k)y$, then $y_k \to y$. By the line segment property, $y_k \in \mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)$. Hence $y \in \overline{\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)}$. Then

$$\overline{C_1} \cap \overline{C_2} \subseteq \overline{\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)} \subseteq \overline{C_1 \cap C_2}.$$

Hence $\overline{C_1 \cap C_2} = \overline{C_1} \cap \overline{C_2}$. Moreover, the closure of $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)$ and $C_1 \cap C_2$ are the same. Hence, they have the same relative interior. Then

$$\mathrm{ri}(C_1 \cap C_2) = \mathrm{ri}(\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)) \subseteq \mathrm{ri}(C_1) \cap \mathrm{ri}(C_2).$$

□

Proposition: Let $B : \mathbb{R}^n \to \mathbb{R}^p$ be an affine mapping and let Ω be a convex subset of $\mathbb{R}^n$. Then

$$B(\mathrm{ri}\ \Omega) = \mathrm{ri}\ B(\Omega).$$

Proof. Let $y \in B(\mathrm{ri}\ \Omega)$, then there exits $x \in \mathrm{ri}\ \Omega$ such that $y = Bx$. By the prolongation lemma, for any $\bar{x} \in \Omega$, there exists $\gamma > 0$ such that $x + \gamma(x - \bar{x}) \in \Omega$. Hence $y + \gamma(y - \bar{y}) = B(x + \gamma(x - \bar{x})) \in B(\Omega)$, where $\bar{y} = B\bar{x}$. Since $\bar{x}$ is arbitrary, by the prolongation lemma again, $y \in \mathrm{ri}\ B(\Omega)$. Hence $B(\mathrm{ri}\ \Omega) \subseteq \mathrm{ri}\ B(\Omega)$.
To show the other direction, we first show that $\overline{B(\Omega)} = \overline{B(\mathrm{ri}\ \Omega)}$. Note that $\overline{\Omega} = \overline{\mathrm{ri}\ \Omega}$, hence we have

$$B(\Omega) \subseteq B(\overline{\Omega}) = B(\overline{\mathrm{ri}\ \Omega}) \subseteq \overline{B(\mathrm{ri}\ \Omega)},$$

where the last inclusion follows from the continuity of B. This shows that $\overline{B(\Omega)} \subseteq \overline{B(\mathrm{ri}\ \Omega)}$. Since $\overline{B(\mathrm{ri}\ \Omega)} \subseteq \overline{B(\Omega)}$, we have $\overline{B(\Omega)} = \overline{B(\mathrm{ri}\ \Omega)}$.
Now since $\overline{B(\Omega)} = \overline{B(\mathrm{ri}\ \Omega)}$, $\mathrm{ri}\ B(\Omega) = \mathrm{ri}\ B(\mathrm{ri}\ \Omega)$. Hence

$$\mathrm{ri}\ B(\Omega) = \mathrm{ri}\ B(\mathrm{ri}\ \Omega) \subseteq B(\mathrm{ri}\ \Omega).$$

□