1 Convex Sets and Functions

1.1 Convex Sets

Definition:(Convex sets) A subset C of $\mathbb{R}^n$ is called *convex* if

$$\lambda x + (1-\lambda)y \in C,\ \forall\ x, y \in C,\ \forall \lambda \in [0,1].$$

Geometrically, it just means that the line segment joining any two points in a convex set C lies in C.

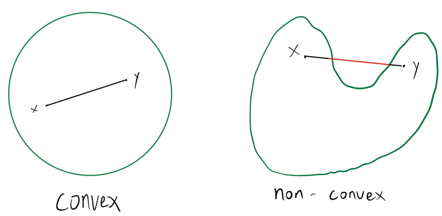

Figure 1: Convex and non-convex set

Definition:(Convex combination) Given $x_1, ..., x_m \in \mathbb{R}^n$, an element in the form $x = \sum_{i=1}^m \lambda_i x_i$, where $\sum_{i=1}^m \lambda_i = 1$ and $\lambda_i \geq 0$ is called a convex combination of $x_1, ..., x_m$.

Proposition: A subset C of $\mathbb{R}^n$ is convex if and only if contains all convex combination of its element.

Proof. Suppose C is convex. We will show by induction that it contains all convex combination $\sum_{i=1}^m \lambda_i x_i$ of its elements.
The case $m = 1, 2$ is trivial, so suppose all convex combination of k elements lies in C, where $k \leq m$. Consider

$$x := \sum_{i=1}^{m+1} \lambda_i x_i, \text{ where } \sum_{i=1}^{m+1} \lambda_i = 1$$

If $\lambda_{m+1} = 1$, then $\lambda_1 = \cdot = \lambda_m = 0$. Then $x \in C$. So assume $\lambda_{m+1} < 1$, then

$$\sum_{i=1}^{m} \lambda_i = 1 - \lambda_{m+1} \text{ and } \sum_{i=1}^{m} \frac{\lambda_i}{1-\lambda_{m+1}} = 1$$

Then $y = \sum_{i=1}^m \frac{\lambda_i}{1-\lambda_{m+1}} x_i \in C$. Hence

$$x = (1-\lambda_{m+1})y + \lambda_{m+1} x_{m+1} \in C$$

The other direction is trivial. □

Proposition: Let C_1 be a convex set of $\mathbb{R}^n$ and let C_2 be a convex set pf $\mathbb{R}^m$. Then the Cartesian product $C_1 \times C_2$ is a convex subset of $\mathbb{R}^n \times \mathbb{R}^m$.

1.1.1 Examples of Convex Sets

(a) Open and closed balls in $\mathbb{R}^n$.

(b) *Hyperplanes*: $\{x : \langle a, x\rangle = b,\ a \in \mathbb{R}^n,\ b \in \mathbb{R}\}$.

(c) *Halfspaces*: $\{x : \langle a, x\rangle \leq b,\ a \in \mathbb{R}^n,\ b \in \mathbb{R}\}$.

(d) *Non-Negative Orthant*: $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geq 0\}$.

(e) Convex cones: C is called a *cone* if $\alpha x \in C, \forall \alpha > 0,\ x \in C$. A cone which is convex is called a *convex cone*.

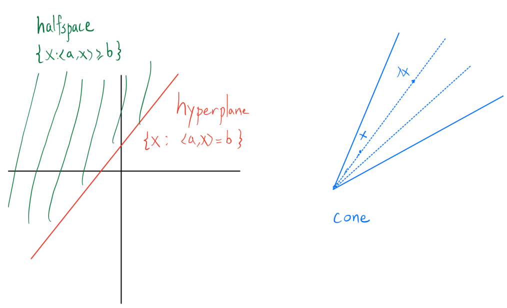

Figure 2: Examples of convex sets

Proposition: Let $\{C_i \mid i \in \mathrm{I}\}$ be a collection of convex sets. Then:

(a) $\cap_{i\in I} C_i$ is convex, where each C_i is convex.

(b) $C_1 + C_2 = \{x + y \mid x \in C_1, y \in C_2\}$ is convex.

(c) λC is convex for any convex sets C and scalar λ. Furthermore, $(\lambda_1 + \lambda_2)C = \lambda_1 C + \lambda_2 C$ for positive λ_1, λ_2.

(d) C^{o}, $\overline{C}$ are convex, i.e. the interior and closure of a convex set are convex.

(e) $T(C)$, $T^{-1}(C)$ are convex, where T is a linear map.

Proof. Parts (a)-(c), (e) follows from the definition (Exercise!). Let's prove (d).
Interior Let $x, y \in C^o$. Then there exists r such that balls with radius r centred at x and y are both inside C.
Suppose $\lambda \in [0,1]$ and $||z|| < r$. By convexity of C, we have,

$$\lambda x + (1-\lambda)y + z = \lambda(x+z) + (1-\lambda)(y+z) \in C$$

Therefore, $\lambda x + (1-\lambda)y \in C^o$. Hence C^o is convex.
Closure Let $x, y \in \overline{C}$. Then there exists sequences $\{x_k\} \subset C, \{y_k\} \subset C$ such that $x_k \to x, y_k \to y$. Suppose $\alpha \in [0,1]$. Then for each k,

$$\lambda x_k + (1-\lambda)y_k \in C$$

But $\lambda x_k + (1-\lambda)y_k \to \lambda x + (1-\lambda)y \in \overline{C}$. Hence, $\overline{C}$ is convex. □

1.2 Convex and Affine Hulls

1.2.1 Convex Hull

Definition:(Convex Hull)
Let X be a subset of $\mathbb{R}^n$. The convex hull of X is defined by

$$\text{conv}(X) := \bigcap \{C |\ C \text{ is convex and } X \subseteq C\}$$

In other words, conv(X) is the smallest convex set containing X.
The next proposition provides a good representation for elements in the convex hull.

Proposition: For any subset X of $\mathbb{R}^n$,

$$\text{conv}(X) = \Big\{ \sum_{i=1}^{m} \lambda_i x_i |\ \sum_{i=1}^{m} \lambda_i = 1,\ \lambda_i \geq 0,\ x_i \in X \Big\}$$

Proof. Let $C = \left\{ \sum_{i=1}^{m} \lambda_i x_i |\ \sum_{i=1}^{m} \lambda_i = 1,\ \lambda_i \geq 0,\ x_i \in X \right\}$. Clearly, $X \subseteq C$.
Next, we check that C is convex.
Let $a = \sum_{i=1}^{p} \alpha_i a_i, b = \sum_{j=1}^{q} \beta_j b_j$ be elements of C, where $a_i, b_i \in C$ with $\alpha_i, \beta_j \geq 0$ and $\sum \alpha_i = \sum \beta_j = 1$. Suppose $\lambda \in [0,1]$, then

$$\lambda a + (1-\lambda)b = \sum_{i=1}^{p} \lambda \alpha_i a_i + \sum_{j=1}^{q} (1-\lambda)\beta_j b_j.$$

Since

$$\sum_{i=1}^{p} \lambda \alpha_i + \sum_{j=1}^{q} (1-\lambda)\beta_j = \lambda \sum_{i=1}^{p} \alpha_i + (1-\lambda) \sum_{j=1}^{q} \beta_j = 1$$

we have $\lambda a + (1-\lambda)b \in C$. Hence, C is convex. Also, conv$(X) \subseteq C$ by the definition of conv(X).
Suppose $a = \sum \lambda_i a_i \in C$. Then since each $a_i \in X \subseteq$ conv(X) and conv(X) is convex, we have $a \in$ conv(X). Therefore, conv(X) $=C$. □

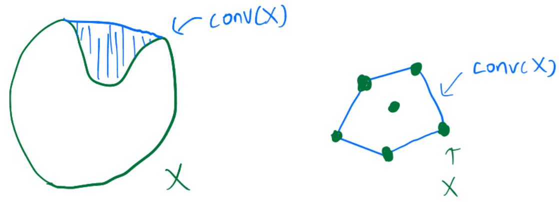

Figure 3: Examples of convex hull

Let $a, b \in \mathbb{R}^n$, define the interval

$$[a, b) := \{\lambda a + (1-\lambda)b | \ \lambda \in (0, 1]\}$$

The intervals $(a, b], (a, b)$ are defined similarly.

Lemma: For a convex set $C \in \mathbb{R}^n$ with nonempty interior, take $a \in C^o$ and $b \in \overline{C}$. Then $[a, b) \subset C^o$.

Proof. Since $b \in \overline{C}$, for any $\epsilon > 0$, we have $b \in C + \epsilon\mathbf{B}$, where $\mathbf{B}$ denotes the closed unit ball centered at 0.
Take $\lambda \in (0, 1]$ and let $x_\lambda := \lambda a + (1-\lambda)b$. Let ϵ be such that $a + \epsilon\frac{2-\lambda}{\lambda}\mathbf{B} \subset C$.

$$\begin{aligned} x_\lambda + \epsilon\mathbf{B} &= \lambda a + (1-\lambda)b + \epsilon\mathbf{B} \\ &\subset \lambda a + (1-\lambda)[C + \epsilon\mathbf{B}] + \epsilon\mathbf{B} \\ &= \lambda a + (1-\lambda)C + (2-\lambda)\epsilon\mathbf{B} \\ &\subset \lambda[a + \epsilon\frac{2-\lambda}{\lambda}\mathbf{B}] + (1-\lambda)C \\ &\subset \lambda C + (1-\lambda)C \subset C \end{aligned}$$

Hence $x_\lambda \in C^o$ and $[a, b) \subset C^o$.

□

1.2.2 Affine Sets and Affine Hull

Given $a, b \in \mathbb{R}^n$, the line connecting them is defined as

$$\mathcal{L}[a, b] := \{\lambda a + (1-\lambda)b | \ \lambda \in \mathbb{R}\}$$

Note that there is no restriction on λ.

Definition:(Affine Set) A subset S of $\mathbb{R}^n$ is *affine* if for any $a, b \in S$, we have $\mathcal{L}[a, b] \subseteq S$.

Definition:(Affine Combination)
Given $x_1, ..., x_m \in \mathbb{R}^n$, an element in the form $x = \sum_{i=1}^m \lambda_i x_i$, where $\sum_{i=1}^m \lambda_i = 1$ is called an affine combination of $x_1, ..., x_m$.

Proposition: A set S is affine if and only if it contains all affine combination of its elements.

Definition:(Affine Hull) The *affine hull* of a set $X \subseteq \mathbb{R}^n$ is

$$\text{aff}(X) := \bigcap \{S| \ S \text{ is affine and } X \subseteq S\}$$

Proposition: For any subset X of $\mathbb{R}^n$,

$$\text{aff}(X) = \{ \sum_{i=1}^m \lambda_i x_i | \ \sum_{i=1}^m \lambda_i = 1, \ x_i \in X\}$$

In fact, an *affine set* $S \subset \mathbb{R}^n$ is of the form $x + V$, where $x \in S$ and V is a vector space called the subspace parallel to S.

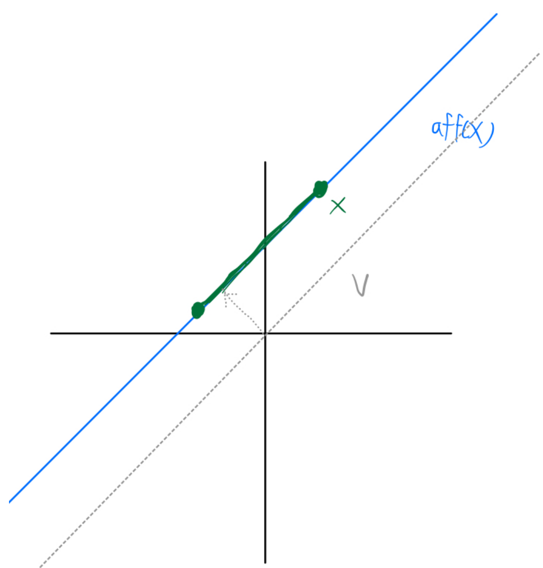

Figure 4: Affine hull and the parallel subspace

Lemma: Let S be nonempty. Then the following are equivalent:

1. S is affine
2. S is of the form $x + V$ for some subspace $V \subset \mathbb{R}^n$ and $x \in S$.

Also, V is unique and equals to $S - S$.

Proof. Suppose S is affine. We first assume $0 \in S$. Let $x \in S$ and $\gamma \in \mathbb{R}$. Since $0 \in S$, we have $\gamma x + (1 - \gamma)0 = \gamma x \in S$. Now, suppose $x, y \in S$. Then $x + y = 2(\frac{1}{2}x + \frac{1}{2}y) \in S$. Hence, S is closed under addition and scalar multiplication. Therefore, $S = 0 + S$ is a linear subspace. If $0 \notin S$, then $0 \in S - x$ for any $x \in S$. So $S - x$ is a linear subspace. Therefore, $S = x + V$.
The other direction is simple, just use the fact that V is a linear subspace.
Now suppose $S = x_1 + V_1 = x_2 + V_2$, where $x_1, x_2 \in S$, V_1, V_2 are linear subspaces. Then $x_1 - x_2 + V_1 = V_2$. Since V_2 is a subspace, $x_1 - x_2 \in V_1$. So $V_2 = x_1 - x_2 + V_1 \subseteq V_1$. Similarly, $V_1 \subseteq V_2$. Therefore V is unique.
Since $S = x + V$, so $V = S - x \subseteq S - S$. Let $u, v \in S$ and $z = u - v$. Then $S - v = V$ by the uniqueness of V. So $z \in S - v = V$ and hence $S - S \subseteq V$. □

Definition:(Dimension of affine and convex sets) The dimension of $\text{aff}(X)$ is defined to be the dimension of the subspace parallel to X. The dimension of a convex set C is defined to be the dimension of $\text{aff}(C)$.