1 Convex and Affine Hulls

Given $x_1, x_2, ..., x_k \in \mathbb{R}^n$, a *convex combination* of $x_1, x_2, ..., x_k$ is a point of the form

$$\sum_{i=1}^{m} \alpha_i x_i \text{ , where } \alpha_i \geq 0 \text{ and } \sum_{i=1}^{m} \alpha_i = 1.$$

An *affine combination* of $x_1, x_2, ..., x_k$ is a point of the form

$$\sum_{i=1}^{m} \alpha_i x_i \text{ , where } \sum_{i=1}^{m} \alpha_i = 1.$$

The *convex hull* of a set X, denoted by $\text{conv}(X)$, is the intersection of all convex sets containing X.
The *affine hull* of a set X, denoted by $\text{aff}(X)$, is the intersection of all affine sets containing X.

It is simple to verify that the $\text{conv}(X)$ is equal to the set of all convex combination of elements in X, while $\text{aff}(X)$ is equal to the set of all affine combination of elements in X.

The cone generated by a set X is the set of all nonnegative combination of elements in X. A nonnegative (positive) combination of $x_1, x_2, ..., x_m$ is of the form

$$\sum_{i=1}^{m} \alpha_i x_i, \text{ where } \alpha_i \geq 0 \ (\alpha_i > 0).$$

2 Caratheodory's Theorem

Theorem: (Caratheodory's Theorem) Let X be a nonempty subset of $\mathbb{R}^n$.

1. Every nonzero vector of $\text{cone}(X)$ can be represented as a positive combination of linearly independent vectors from X.

2. Every vector from $\text{conv}(X)$ can be represented as a convex combination of at most $n+1$ vectors from X.

Proof. 1) Let $x \in \text{cone}(X)$ and $x \neq 0$. Suppose m is the smallest integer such that x is of the form $\sum_{i=1}^{m} \alpha_i x_i$, where $\alpha_i > 0$ and $x_i \in X$.
Suppose that x_i are not linearly independent. Therefore, there exist λ_i with at least one λ_i positive, such that $\sum_{i=1}^{m} \lambda_i x_i = 0$.
Consider $\overline{\gamma}$, the largest γ such that $\alpha - \gamma\lambda_i \geq 0$ for all i.
Then $\sum_{i=1}^{m} (\alpha_i - \overline{\gamma}\lambda)x_i$ is a representation of x as a positive combination of less than m vectors, contradiction. Hence, x_i are linearly independent.

2) Consider $Y = \{(x, 1) : x \in X\}$. Let $x \in \text{conv}(X)$. Then $x = \sum_{i=1}^{m} \alpha_i x_i$, where $\sum_{i=1}^{m} \alpha_i = 1$, so $(x, 1) \in \text{cone}(Y)$.
By 1), $(x, 1) = \sum_{i=1}^{l} \alpha_i'(x_i, 1)$, where $\alpha_i > 0$. Also, $(x_1, 1), ..., (x_l, 1)$ are linearly independent vectors in $\mathbb{R}^{n+1}$(at most $n+1$). Hence, $x = \sum_{i=1}^{l} \alpha_i' x_i$, $\sum_{i=1}^{m} \alpha_i' = 1$ □

I am sorry for the confusion in the tutorial, I hope this notes will be more clear to you.