THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2018)
HW5 Solution

1. (P.215 Q2)

h is clearly bounded on $[0,1]$. Applying Theorem 1.8 of the lecture note 1 P.3, it suffices to show that there exists $\epsilon_0 > 0$ such that for all partition $P := a = x_0 < x_1 < ... < x_n = b$ on $[0,1]$, we have

$$U(h,P) - L(h,P) \geq \epsilon_0$$

Let $\epsilon_0 = 1$, then for all partition $P := a = x_0 < x_1 < ... < x_n = b$ on $[a,b]$. For each $1 \leq i \leq n$, since $\mathbb{Q} \cap [0,1]$ is dense in $[0,1]$, there exists $(y_m^{(i)})_{m=1}^{\infty} \subseteq \mathbb{Q} \cap [x_{i-1}, x_i]$ such that $y_m^{(i)} \to x_i$ as $m \to \infty$. Since $h(x) \leq x_i + 1$ on $[x_{i-1}, x_i]$ by definition, we have

$$M_i(h,P) = x_i + 1$$

On the other hand, since $(\mathbb{R}\backslash\mathbb{Q}) \cap [0,1]$ is dense in $[0,1]$, $(\mathbb{R}\backslash\mathbb{Q}) \cap [x_{i-1}, x_i] \neq \phi$, and hence $h(z_i) = 0$ for some $z_i \in (\mathbb{R}\backslash\mathbb{Q}) \cap [x_{i-1}, x_i]$. Since $h(x) \geq 0$ on $[x_{i-1}, x_i]$ by definition, we have

$$m_i(h,P) = 0$$

Therefore,

$$\begin{aligned}
U(h,P) - L(h,P) &= \sum_{i=1}^{n} \omega_i(h,P)\Delta x_i \\
&= \sum_{i=1}^{n} (x_i + 1)(x_i - x_{i-1}) \\
&\geq \sum_{i=1}^{n} (x_i - x_{i-1}) \\
&= x_n - x_0 = 1 = \epsilon_0
\end{aligned}$$

Therefore, h is not integrable on $[0,1]$.

2. (P.215 Q10)

Define $h(x) = f(x) - g(x)$ on $[a,b]$, then h is continuous on $[a,b]$ (and hence Riemann integrable by Prop. 1.11 in Lecture note 1 P.5) and $\int_a^b h = \int_a^b f - \int_a^b g$ (by Prop. 1.7 of lecture note 1 P.3)= 0.

Now we prove by contradiction: suppose on the contrary for all $c \in [a,b]$, $f(c) \neq g(c)$, i.e. $h(c) \neq 0$. Since h is continuous on $[a,b]$, by Intermediate Value Theorem, either (i) $h(x) > 0$ for all $x \in [a,b]$ or (ii) $h(x) < 0$ all $x \in [a,b]$.

Case (i): applying the result of Q8 (since h is non-negative on $[a, b]$ and $\int_a^b h = 0$), we must have $h(x) = 0$ for all $x \in [a, b]$, which is a contradiction.

Case (ii) Let $k(x) = -h(x)$ on $[a, b]$. Then apply case (i) to $k(x)$ to derive a contradiction.

Therefore, both leads to contradiction. Hence there exists $c \in [a, b]$ such that $f(c) = g(c)$.

3. (P.215 Q11)

   We first show that $f \in R[a, b]$: Since f is bounded, by Prop. 1.8 of the Lecture note, it suffices to show that for all $\epsilon > 0$, there exists a partition $P := a = x_0 < x_1 < ... < x_n = b$ on $[a, b]$, we have

$$U(f, P) - L(f, P) < \epsilon$$

   Let $\epsilon > 0$ be given, choose $c = a + \delta$, where $0 < \delta < \min\{\frac{\epsilon}{4M + 1}, b - a\}$

   Then $c \in (a, b)$, and hence by the integrability of f on $[c, b]$, there exists a partition $P' := c = x_0 < x_1 < ... < x_n = b$ on $[c, b]$ such that

$$U(f, P') - L(f, P') < \frac{\epsilon}{2}$$

   Define a partition P on $[a, b]$ by $P := a < c < x_1 < ... < x_n = b$. Then

$$\begin{aligned} U(f, P) - L(f, P) &= (\sup_{[a,c]} f - \inf_{[a,c]} f)(c - a) + U(f, P') - L(f, P') \\ &< 2M \cdot \frac{\epsilon}{4M + 1} + \frac{\epsilon}{2} \\ &< \epsilon \end{aligned}$$

   Since $\epsilon > 0$ is arbitrary, $f \in R[a, b]$.

   Then we claim that $\int_c^b f \to \int_a^b f$ as $c \to a^-$: Given $\epsilon > 0$, choose $\delta = \min\{\frac{\epsilon}{M + 1}, b - a\}$. Then for all $a < c < a + \delta$, since $f \in R[a, b]$ and $f|_{[c,b]} \in R[c, b]$, by Prop. 1.13 of the note,

$$|\int_c^b f - \int_a^b f| = |\int_a^c f|$$

   By Prop. 1.12 (ii), $|\int_a^c f| \le \int_a^c |f| \le M(c - a) < M \cdot \frac{\epsilon}{M + 1} < \epsilon$
   Therefore, for all $a < c < a + \delta$, $|\int_c^b f - \int_a^b f| < \epsilon$. This shows $\int_c^b f \to \int_a^b f$ as $c \to a^-$.

4. (P.215 Q18)
   Since $[a, b]$ is compact, there exists $z \in [a, b]$ s.t. $f(z) = \sup\{f(x) : x \in [a, b]\} := S$. Let $\epsilon > 0$, there exists $\delta > 0$ s.t. $|f(x) - M| \le \epsilon$ for all $x \in [a, b] \cap V_\delta(z)$. Then,

$$(S - \epsilon)(2\delta)^{1/n} \le (\int_{[a,b]\cap V_\delta(z)} f^n)^{1/n} \le M_n \le S(b - a)^{1/n}$$

   By sandwich theorem, we have

$$S - \epsilon \le \lim_{n\to\infty} M_n \le S.$$

   Since, ϵ is arbitrary, the proof is done.