MATH4010

# Tutorial 8

**Proposition 8.6** If $\|\cdot\|$ and $\|\ \|'$ both are complete norms on $X$ such that $\|\cdot\| \le c\|\cdot\|'$ for some $c>0$, then these two norms are equivalent.

**Ex1.** Use the above proposition to prove $(C[0,1], \|\cdot\|_1)$ is not a Banach space, where $\|f\|_1 = \int_0^1 |f(t)|\,dt$

**Pf.** Suppose that $(C[0,1], \|\cdot\|_1)$ is a Banach space. Since $\|\cdot\|_\infty$ $\left(\|f\|_\infty = \max\limits_{t\in[0,1]} |f(t)|\right)$ is also a complete norm and $\|f\|_1 \le \|f\|_\infty$ $\forall f \in C[0,1]$, then it follows from the open mapping thm that $\|f\|_1 \ge c\|f\|_\infty$ for some $c>0$. However, that contradicts to the following example: $f_n(t) = \begin{cases} -\frac{n^2}{2}t + n & 0 \le t \le \frac{2}{n} \\ 0 & \text{otherwise} \end{cases} \in C[0,1]$,

$\|f_n\|_1 = 1$, $\|f_n\|_\infty = n$. $\square$

**Ex2.** Let $T: X \to Y$ and $S: X \to Z$ be linear, bounded operators on Banach spaces

(a) If $M$ is a closed linear subspace of $\ker T$, then $x+M \mapsto Tx$ is well-defined, linear & continuous.

(b) If $S$ is onto and $Sx=0 \Rightarrow Tx=0$, then $Sx \to Tx$ is a well-defined operator in $B(Z,Y)$.

Pf. (a) Omitted.

(b) Define the mapping $R: X/\ker S \to Y$ as $x+\ker S \mapsto Tx$. Then the conclusion of (a) implies that $R$ is well-defined linear bounded operator since $\ker S$ is a closed subspace of $\ker T$. Furthermore, since $S$ is onto, then it follows from the Open Mapping Theorem that

$$S_0: X/\ker S \to Z,\quad x+\ker S \mapsto Sx$$

is an isomorphism.

So the mapping $Sx \mapsto Tx$, which is exactly $R \circ S_0^{-1}$, is a well-defined operator in $B(Z,Y)$. $\square$

Ex3. For a normed space $(X, \|\cdot\|)$, prove that $\|\cdot\|$ is induce by an inner product iff it satisfies the parallelogram law.

Pf. $\Rightarrow$ Omitted

$\Leftarrow$ Suppose that $\|x+y\|^2+\|x-y\|^2 = 2(\|x\|^2+\|y\|^2)$, $\forall x,y \in X$.

Then define $\langle\cdot,\cdot\rangle: X \times X \to \mathbb{C}$

$$\langle x,y\rangle \mapsto \frac{1}{4}\left(\|x+y\|^2-\|x-y\|^2+i\left(\|x+iy\|^2-\|x-iy\|^2\right)\right)$$

It suffices to prove that $\|x\|^2=\langle x,x\rangle$ and $\langle\cdot,\cdot\rangle$ is an inner product.

① $\langle x,x\rangle = \frac{1}{4}\left[\|2x\|^2 + i\left(\|x+ix\|^2-\|x-ix\|^2\right)\right] = \|x\|^2 \geq 0$.

② $$\overline{\langle x,y\rangle} = \frac{1}{4}\left[\|x+y\|^2-\|x-y\|^2 - i\left(\|x+iy\|^2-\|x-iy\|^2\right)\right]$$
$$= \frac{1}{4}\left[\|x+y\|^2-\|x-y\|^2 - i\left(\|y-ix\|^2-\|y+ix\|^2\right)\right] = \langle y,x\rangle$$

③ The parallelogram implies that
$$\|x+y+z\|^2+\|x-y+z\|^2 = 2\|x+z\|^2+2\|y\|^2.$$
$$\Rightarrow \begin{cases} \|x+y+z\|^2 = 2\|x+z\|^2+2\|y\|^2-\|x-y+z\|^2 \\ \|x+y+z\|^2 = 2\|y+z\|^2+2\|x\|^2-\|y-x+z\|^2 \end{cases}$$
$$\Rightarrow \|x+y+z\|^2 = \|x\|^2+\|y\|^2+\|x+z\|^2+\|y+z\|^2-\|x-y+z\|^2-\|y-x+z\|^2$$
$$\|x+y-z\|^2 = \|x\|^2+\|y\|^2+\|x-z\|^2+\|y-z\|^2-\|x-y-z\|^2-\|y-x-z\|^2$$
$$\Rightarrow \langle x+y,z\rangle = \frac{1}{4}\left[\|x+z\|^2-\|x-z\|^2+\|y+z\|^2-\|y-z\|^2 + i\left(\|x+iz\|^2-\|x-iz\|^2+\|y+iz\|^2-\|y-iz\|^2\right)\right]$$
$$= \langle x,z\rangle+\langle y,z\rangle.$$

④ By induction, we have $\langle \alpha x,y\rangle = \alpha\langle x,y\rangle$ (*) for all $\alpha\in\mathbb{N}$.
Since $\langle -x,y\rangle+\langle x,y\rangle=0$, $\langle -x,y\rangle = -\langle x,y\rangle$. Then the above equality holds for all $\alpha\in\mathbb{Z}$. Now if $\alpha=\frac{p}{q}$, $p,q\in\mathbb{Z}$, $q\neq 0$, then $q\langle\alpha x,y\rangle = q\langle p\frac{x}{q},y\rangle = p\langle q\frac{x}{q},y\rangle = p\langle x,y\rangle$.
Dividing it by $q$ gives that the above equality (*) holds for all $\alpha\in\mathbb{Q}$.
We have just seen that for fixed $x,y$ the continuous function $t\mapsto\frac{1}{t}\langle tx,y\rangle$ defined on $\mathbb{R}\setminus\{0\}$ is equal to $\langle x,y\rangle$ for all $t\in\mathbb{Q}\setminus\{0\}$, thus is also equal to $\langle x,y\rangle$ for all $t\in\mathbb{R}\setminus\{0\}$.

$$\langle ix, y\rangle = \frac{1}{4}\left[ \|ix+y\|^2 - \|ix-y\|^2 + i\left( \|ix+iy\|^2 - \|ix-iy\|^2 \right) \right]$$

$$= i\,\frac{1}{4}\left[ \|x+y\|^2 - \|x-y\|^2 - i\left( \|ix+y\|^2 - \|ix-y\|^2 \right) \right]$$

$$= i\langle x. y\rangle .$$

$$\Rightarrow \langle \alpha x, y\rangle = \alpha\langle x. y\rangle \text{ for all } \alpha \in \mathbb{C}.$$

Combining ①. ②. ③, $\|\cdot\|$ is induced by the inner product $\langle \cdot, \cdot \rangle$. $\square$